Decorrelation of deformation models via varifolds

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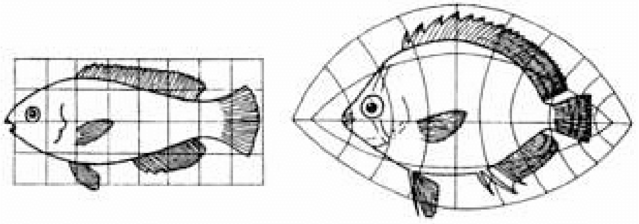
Introduction to computational anatomy

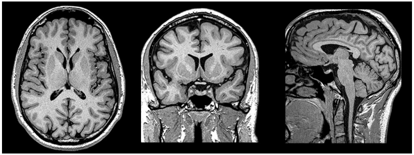
Main idea : Modelize and analyze the variability of biological shapes.

Computational anatomy introduced by the biologist D'Arcy Thompson (1917) in "On Growth and Form"

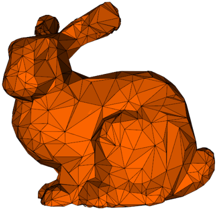


On Growth and Form - D'Arcy Thompson

Different types of data

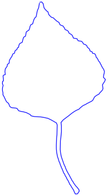


Images



Meshed surfaces

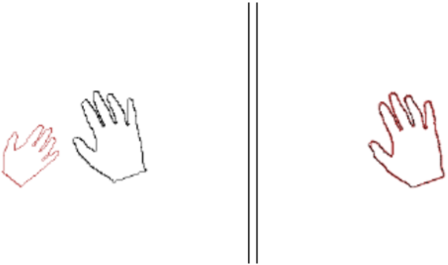
Landmarks



Curves

Registration/Matching

Given two objects $q^{(0)}, q^{(1)}$, we want to find the best diffeomorphism that transform the source $q^{(0)}$ into the target $q^{(1)}$.



Robust Rigid Shape Registration Method Using a Level Set Formulation - M.Al-Huseiny, S.Mahmoodi, M.Nixon

Large Deformation Diffeomorphic Metric Mapping[1]

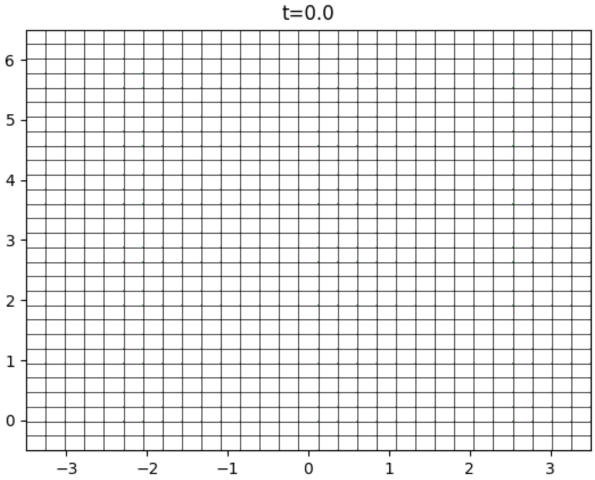
Theorem

Let $v \in L^2([0,1], V)$ *be a time-varying vector field with* $V \hookrightarrow \mathcal{C}_0^2(\mathbb{R}^d, \mathbb{R}^d)$. *The flow of diffeomorphism* φ^v *generated by* v *is the unique solution of the following system :*

$$\begin{cases} \dot{\varphi}_t^v & = & v_t \circ \varphi_t^v \\ \varphi_0^v & = & \mathrm{Id} \end{cases}$$

[1] An Infinite Dimensional Group Approach for Physics based Models in Pattern Recognition, Trouvé, '95

Diffeomorphism generated by a vector field



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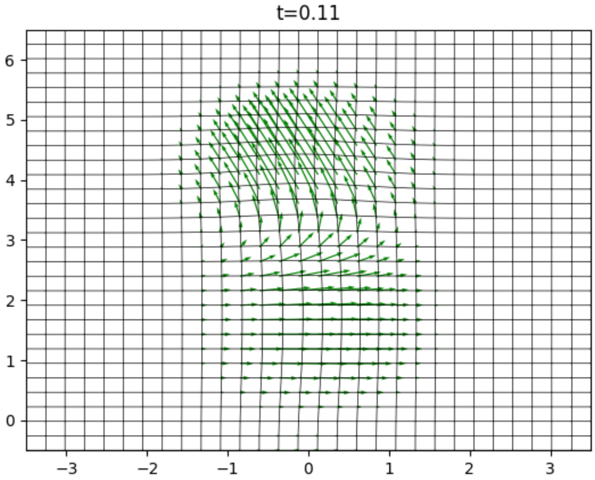

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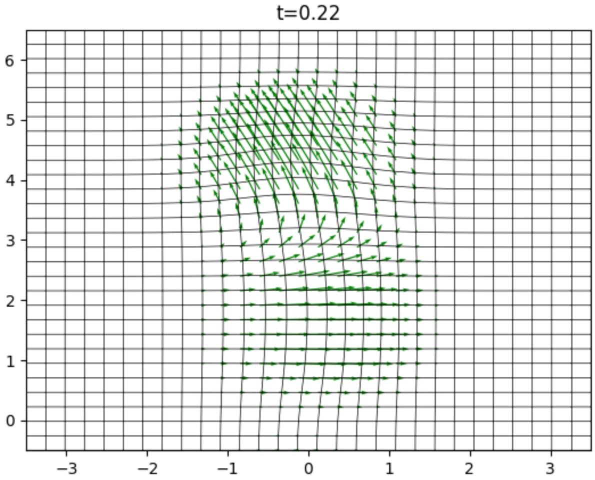

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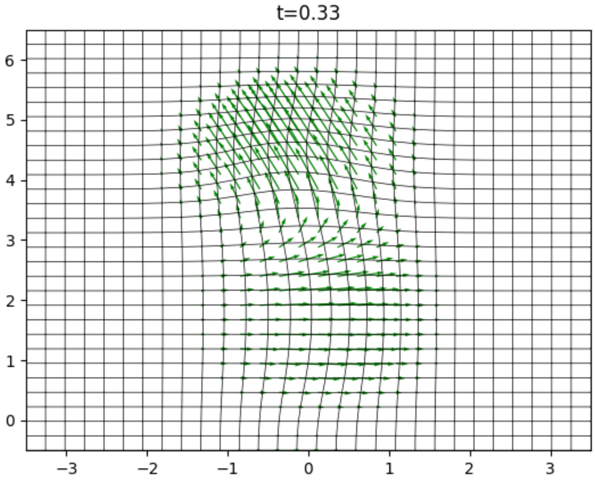

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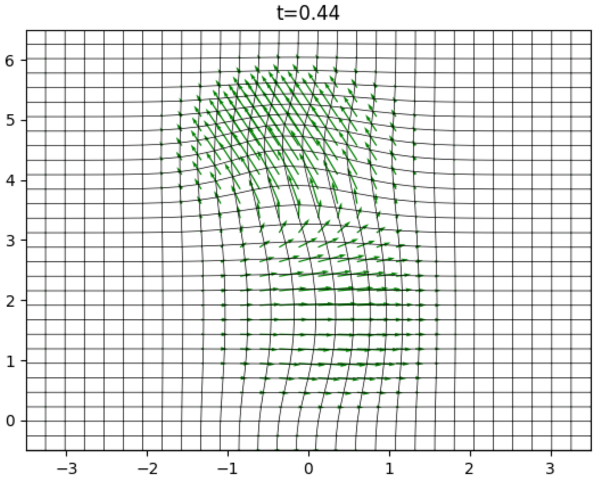
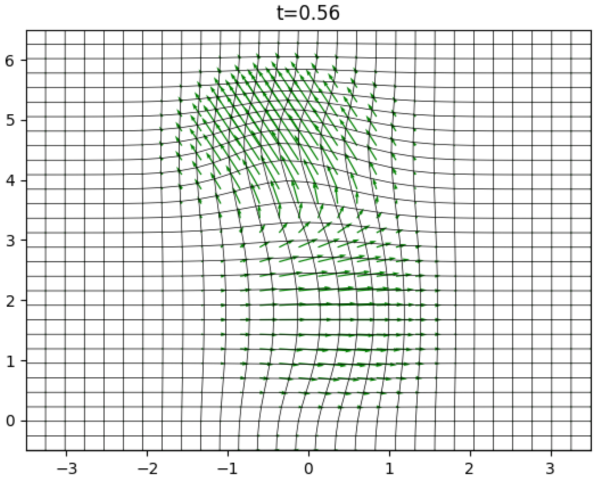


Diffeomorphism generated by a vector field



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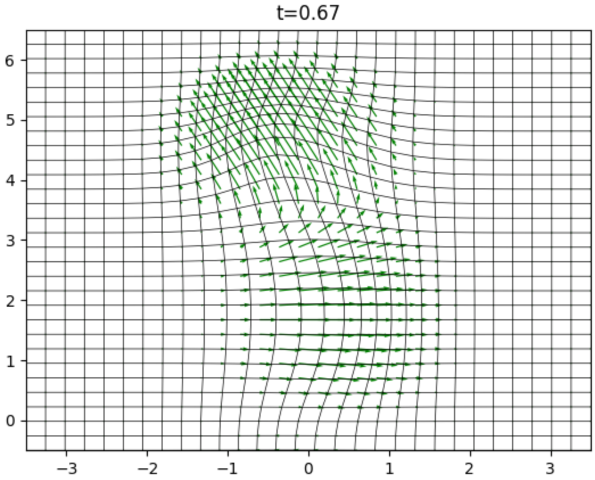

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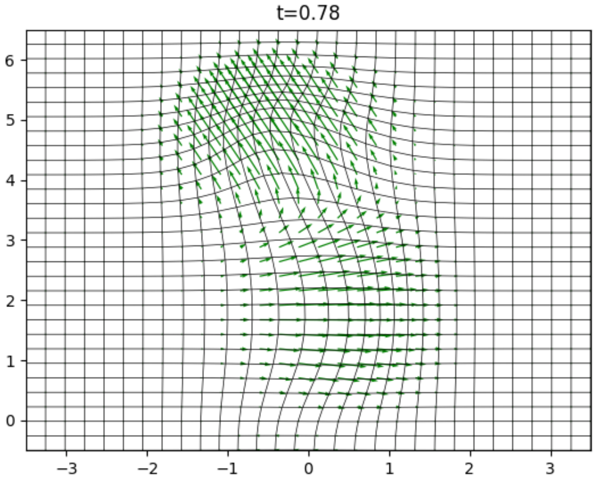

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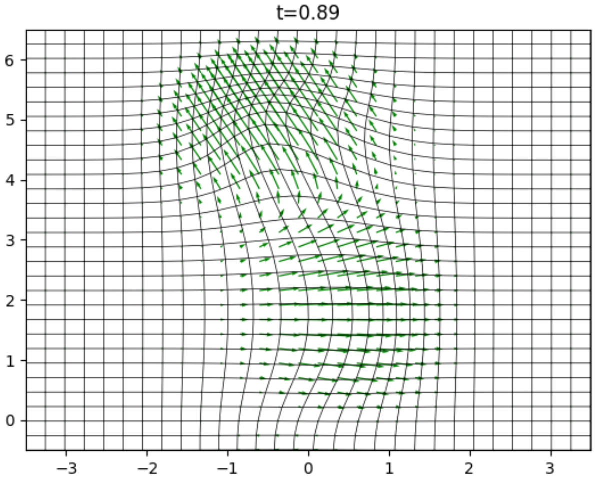
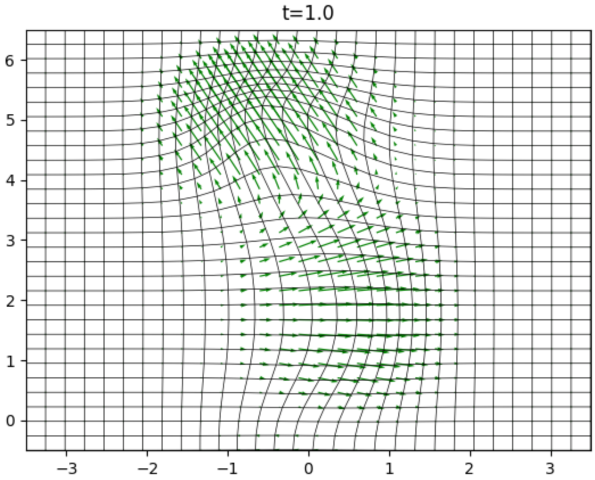


Diffeomorphism generated by a vector field



Shape dynamic

The deformed source is defined by the action of a diffeomorphism on the source

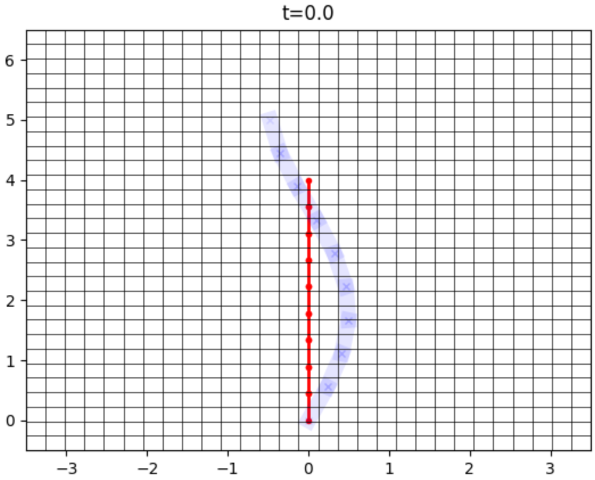
$$q_t = \varphi_t \cdot q^{(0)}$$

From the dynamic of the diffeomorphism, we deduce a dynamic of a deformed shape[2] is given by

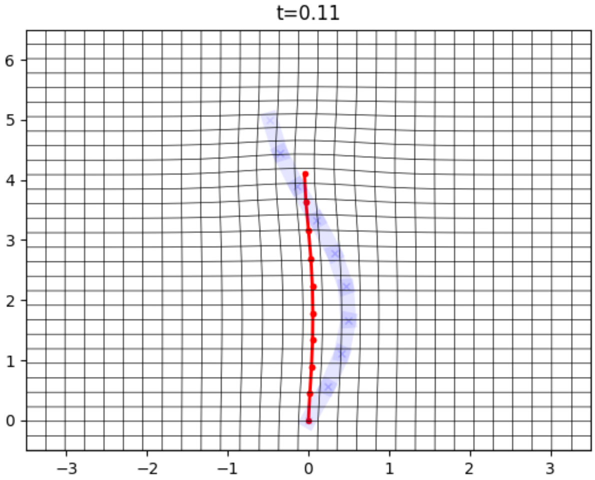
$$\begin{cases} \dot{q}_t &= v_t \cdot q_t \\ q_0 &= q^{(0)} \end{cases}$$

[2] Arguillere's PhD thesis, '14

Shape deformation



Shape deformation



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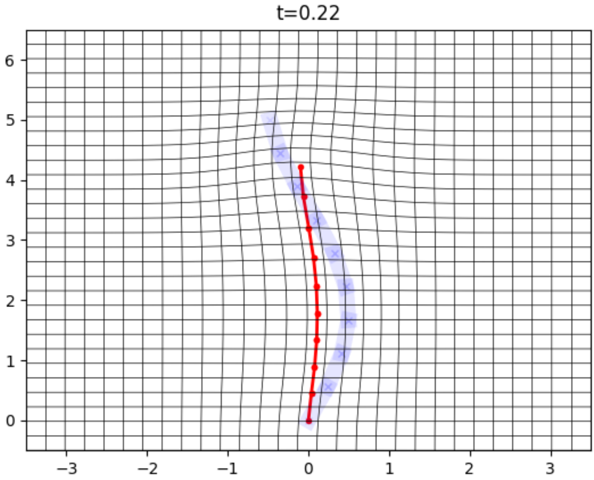

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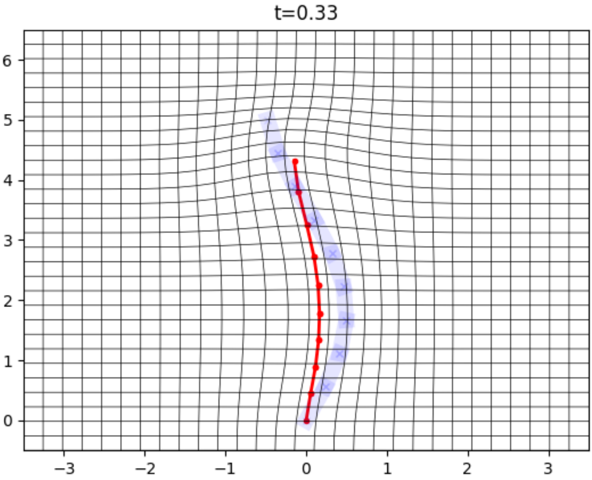
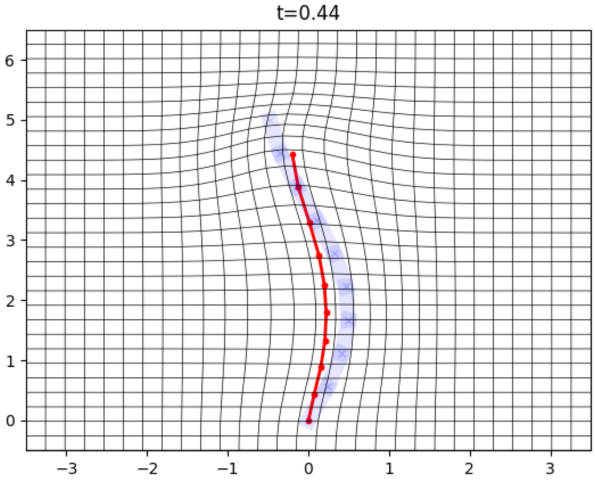
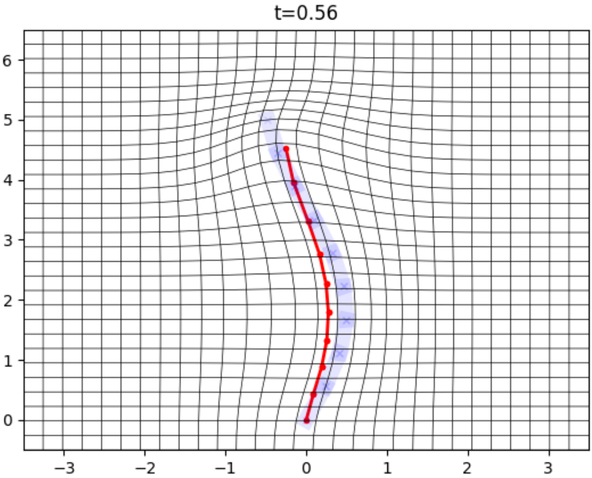


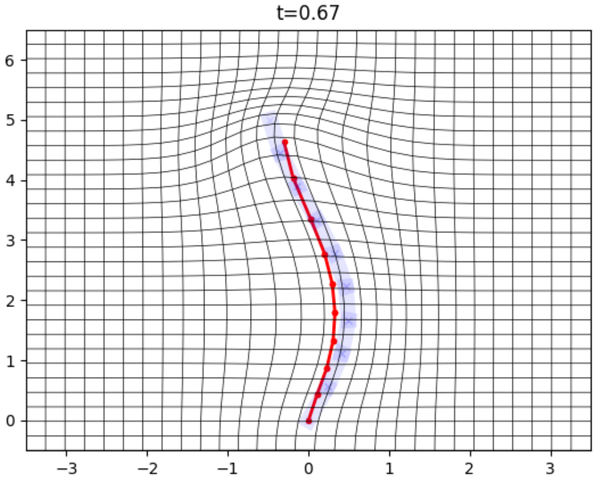
Shape deformation



Shape deformation



Shape deformation



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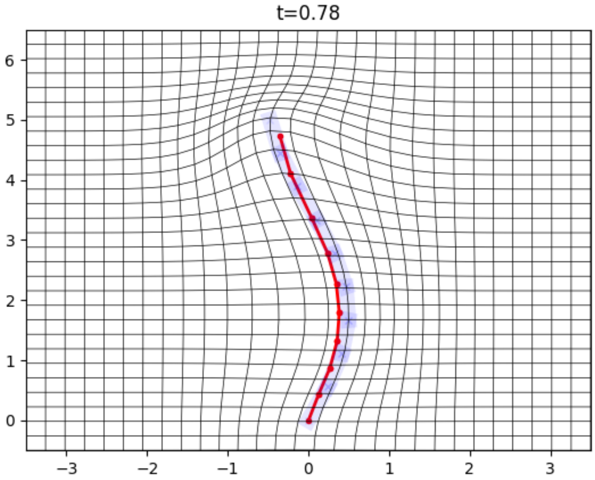
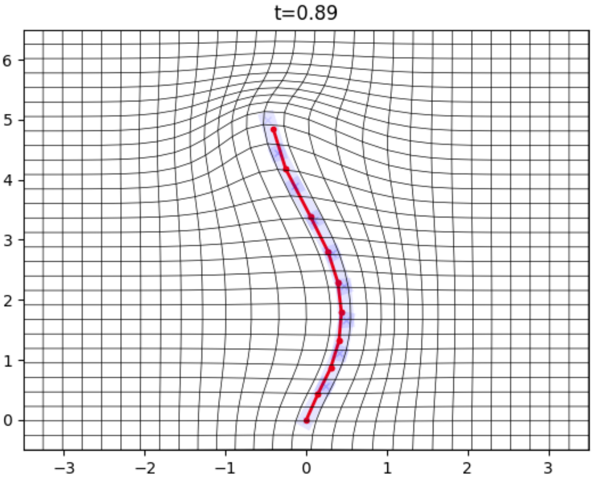
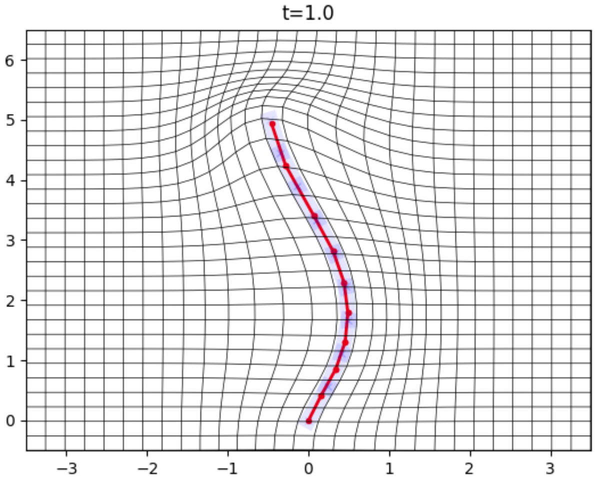


Shape deformation



Shape deformation

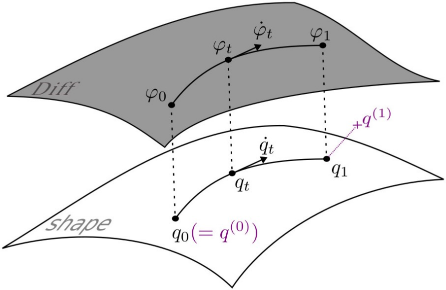


LDDMM : Inexact matching shapes

Finding the best diffeomorphism is an energy minimization problem :

$$\min_v E(v) = \int_0^1 \frac{1}{2}|v_t|_V^2 \; dt + D(\varphi_1 \cdot q^{(0)}, q^{(1)})$$

$$\text{subject to } \begin{cases} \dot{q}_t & = & v_t \cdot q_t \\ q_0 & = & q^{(0)} \end{cases}$$



Coupling of two types of deformations

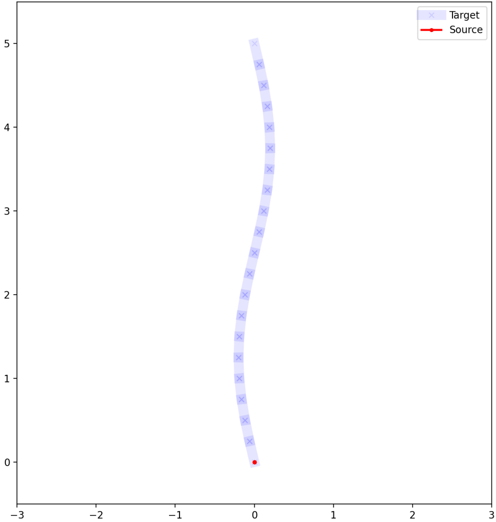
The dynamic of a shape deformed by two vector fields $v \in V, w \in W$ is given by

$$\begin{cases} \dot{q}_t & = & v_t \cdot q_t + w_t \cdot q_t \\ q_0 & = & q^{(0)} \end{cases}$$

and the energy minimization problem associated to this dynamic is

$$\min_{v,w} E(v,w) = \int_0^1 \frac{1}{2}|v_t|_V^2 + \frac{1}{2}|w_t|_W^2 dt + \mathcal{A}(q_1)$$

Correlation problem



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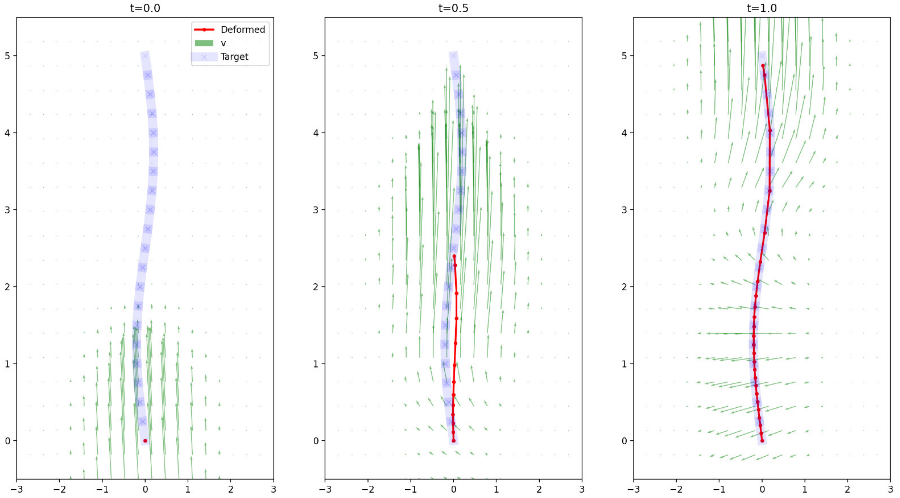
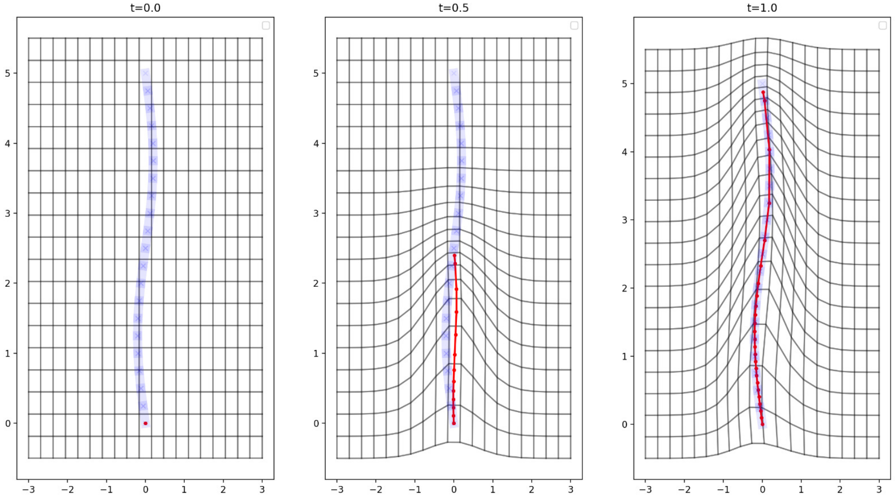


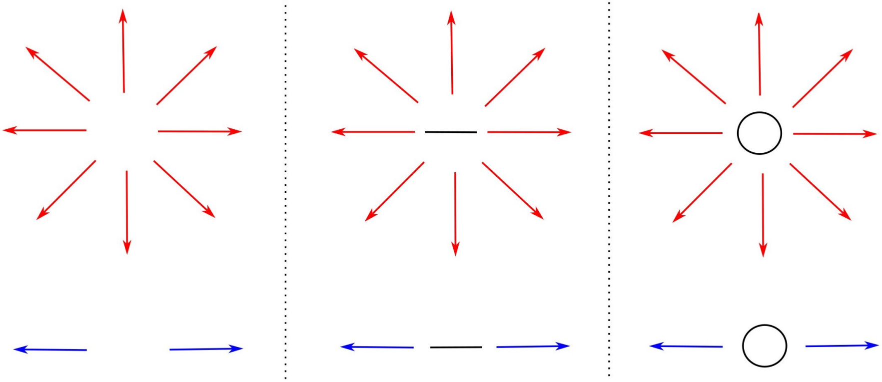
Correlation problem



Correlation

- We want to decorrelate LDDMM and a growth translation in order to have a better interpretation and a more accurate model.
- More generally : How do we decorrelate the actions of two spaces of vector fields on a same curve ?

Intuition of the decorrelation



Varifold

A varifold is a way to represent curves with invariance to reparametrization.

Definition

A varifold is a continuous linear form on $W = \{\omega : \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}\}$.
The varifold μ_q associated to the curve q is defined by :

$$\mu_q(\omega) = \int_q \omega(x, \overrightarrow{t}(x))\, dx$$

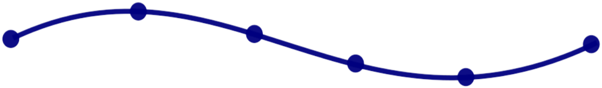
where $\overrightarrow{t}$ represents the tangential data

Discrete varifold

Proposition

A discrete curve can be modeled by a varifold

$$\mu_q(\omega) = \sum_{i=1}^{n} \ell_i \; \omega(c_i, \vec{t_i}) \text{ where } \begin{cases} c_i & = & \frac{q_i + q_{i+1}}{2} \\ \vec{t_i} & = & \frac{q_{i+1} - q_i}{\|q_{i+1} - q_i\|} \end{cases}$$



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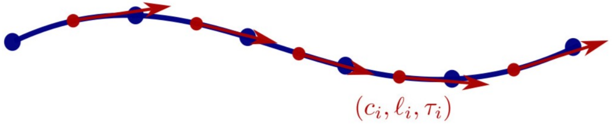

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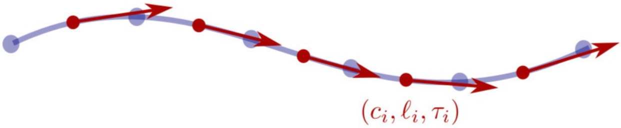

Action of a diffeomorphism on a varifold

Proposition

The action of a diffeomorphism on a varifold is defined by

$$(\phi_* \mu_q)(\omega) = \mu_{\phi(q)}(\omega) = \sum_{i=1}^{n} \ell_i^\phi \ \omega(c_i^\phi, \vec{t_i^\phi})$$

where $c_i^\phi = \frac{\phi(q_{i+1})+\phi(q_i)}{2}$ and $\vec{t_i^\phi} = \frac{\phi(q_{i+1})-\phi(q_i)}{\|\phi(q_{i+1})-\phi(q_i)\|}$

Speed of a varifold induced by a vector field

Proposition

Let $t \mapsto \phi_t$ be a flow of diffeomorphism such that $\phi_0 = \mathrm{Id}$ and $\dot{\phi}_t|_{t=0} = v$

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \mu_{\phi_t(q)}(\omega) &= \sum_{i=1}^{n} \frac{\langle v(q_{i+1}) - v(q_i), q_{i+1} - q_i \rangle}{\ell_i} \, \omega(c_i, \vec{t_i}) \\ &+ \ell_i \left(\partial_x \omega(c_i, \vec{t_i}) \Big| v(c_i) \right) \\ &+ \ell_i \left(\partial_V \omega(c_i, \vec{t_i}) \Big| \nabla^\perp v(q_i) \right) \end{aligned}$$

where $\nabla^\perp v(q_i) = \frac{v(q_{i+1}) - v(q_i)}{\ell_i} - \left(\frac{v(q_{i+1}) - v(q_i)}{\ell_i} \cdot \vec{t_i} \right) \vec{t_i}$

In the following, we will denote $\frac{d}{dt}\big|_{t=0} \mu_{\phi_t(q)}$ as $\delta\mu_q(v)$

Speed of a varifold induced by a vector field

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where $\nabla^{\perp} v(q_i) = \frac{v(q_{i+1}) - v(q_i)}{\ell_i} - \left(\frac{v(q_{i+1}) - v(q_i)}{\ell_i} \cdot \vec{t_i}\right) \vec{t_i}$

Speed of a varifold induced by a vector field

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&+ \ \ell_i \ \overbrace{\left(\partial_x \omega(c_i, \vec{t_i}) \middle| v(c_i)\right)}^{\text{variation w.r.t to the position}} \\
&+ \ \ell_i \ \overbrace{\left(\partial_V \omega(c_i, \vec{t_i}) \middle| \nabla^{\perp} v(q_i)\right)}^{\text{variation w.r.t to the tangent}}
\end{aligned}$$

where $\nabla^{\perp} v(q_i) = \frac{v(q_{i+1})-v(q_i)}{\ell_i} - \left(\frac{v(q_{i+1})-v(q_i)}{\ell_i} \cdot \vec{t_i}\right)\vec{t_i}$

Correlation

We define the correlation between a vector field $v_1 \in V_1$ and a space of vector fields V_2 by

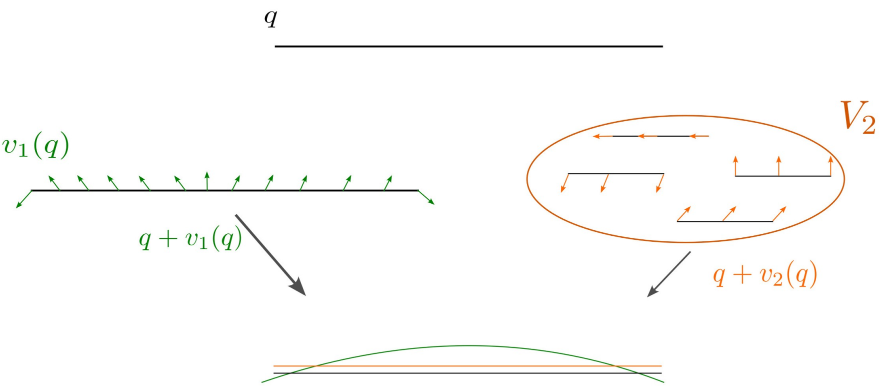
$$\mathrm{Corr}_{\mathrm{q}}(v_1, V_2) = \|v_2^*\|_{V_2}$$

where

$$v_2^* = \mathrm{argmin}_{v_2 \in V_2} \|\delta\mu_q(v_1) - \delta\mu_q(v_2)\|^2$$

.

Correlation



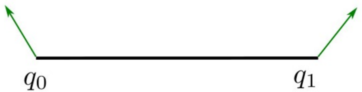
Correlation LDDMM/Translation

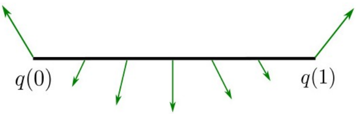
$$\min_{v,\tau} E(v,\tau) = \int_0^1 \frac{1}{2}|v_t|_V^2 + \frac{1}{2}\|\tau_t\|_{\mathbb{R}^d}^2 dt + \int_0^1 \mathrm{Corr}_{q_t}(v_t,\mathbb{R}^d)^2 dt + \mathcal{A}(q_1)$$

where $\mathrm{Corr}_{q_t}(v_t,\mathbb{R}^d) = \|u^*(v_t,q_t)\|_{\mathbb{R}^d}$

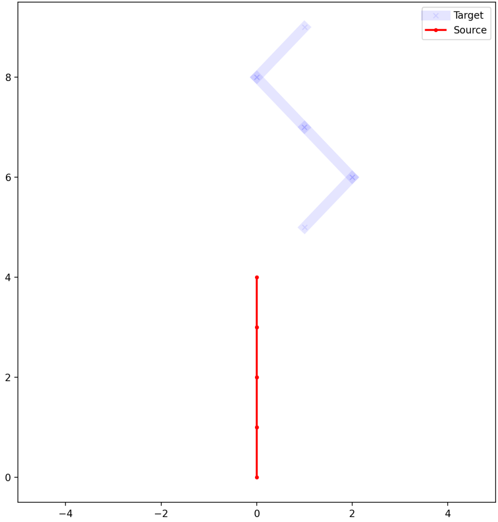
Application to a shape q representing a straight rod :

	Discrete formula (2 pts)	Continuous formula
Curve q	$q_0 = (0,0), q_1 = (L,0)$	$q : s \in [0,1] \mapsto L(s,0)$
Closest translation u^*	$\frac{v(q_0)+v(q_1)}{2}$	$\left(\frac{v_x(0)+v_x(1)}{2}, \int_0^1 \alpha(s) v_y(s) ds\right)$

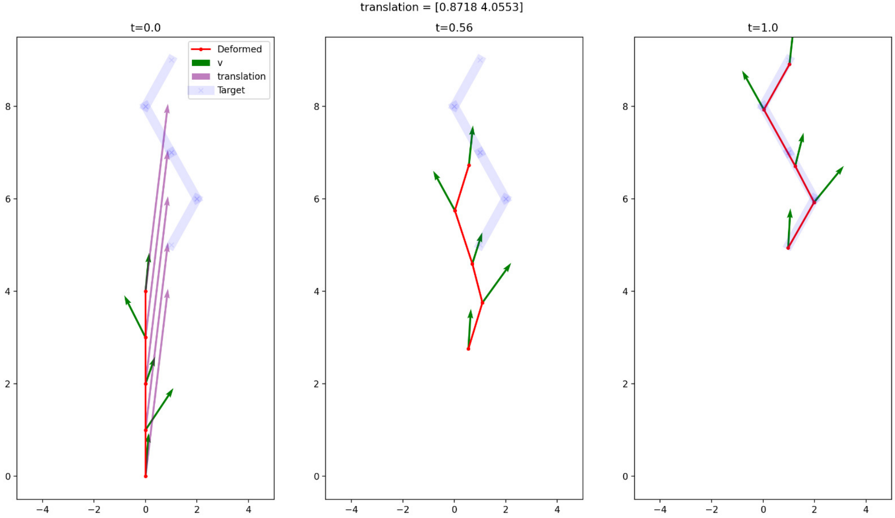




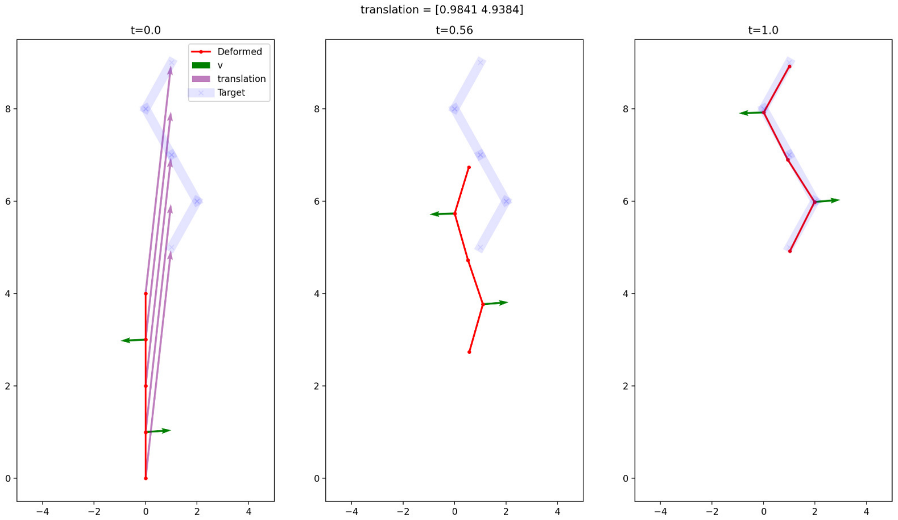
Toy example



Toy example : Without decorrelation



Toy example : With decorrelation



The End

Thank you for your attention !