Decorrelation of vector fields with speed of varifolds

Groupe de travail des éphémères - MAP5

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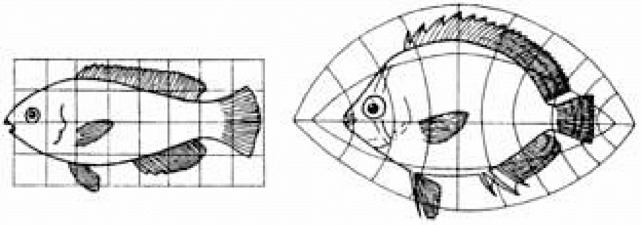
Introduction to computational anatomy

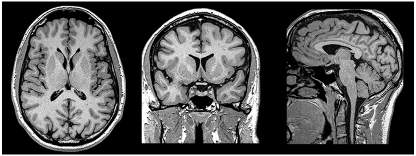
Main idea : Modelize and analyze the variability of biological shapes.

Computational anatomy introduced by the biologist D'Arcy Thompson (1917) in "On Growth and Form"

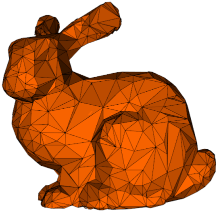


On Growth and Form - D'Arcy Thompson

Different types of data

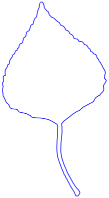


Images



Meshed surfaces

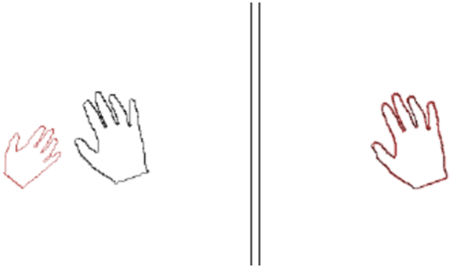
Landmarks



Curves

Registration/Matching

Given two objects $q^{(0)}, q^{(1)}$, we want to find the best diffeomorphism that transform the source $q^{(0)}$ into the target $q^{(1)}$.



Robust Rigid Shape Registration Method Using a Level Set Formulation - M.Al-Huseiny, S.Mahmoodi, M.Nixon

Large Deformation Diffeomorphic Metric Mapping[1]

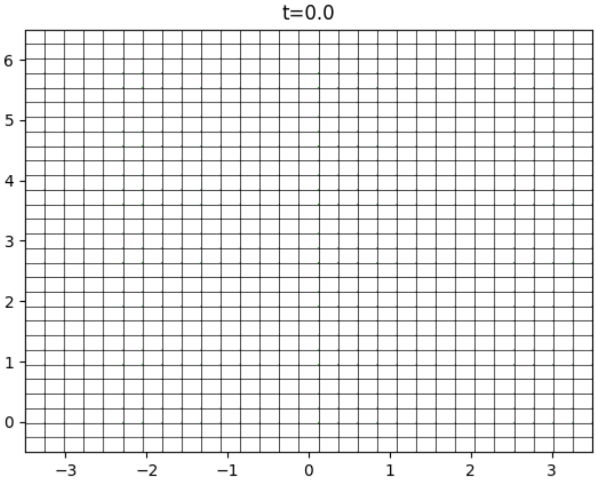
Theorem

Let $v \in L^2([0,1], V)$ *be a time-varying vector field with* $V \hookrightarrow \mathcal{C}_0^2(\mathbb{R}^d, \mathbb{R}^d)$.
The flow of diffeomorphism φ^v *generated by* v *is the unique solution of :*

$$\begin{cases} \dot{\varphi}_t^v & = & v_t \circ \varphi_t^v \\ \varphi_0^v & = & \mathrm{id} \end{cases}$$

[1] Beg, Miller, Trouvé, Younes 2005

Diffeomorphism generated by a vector field



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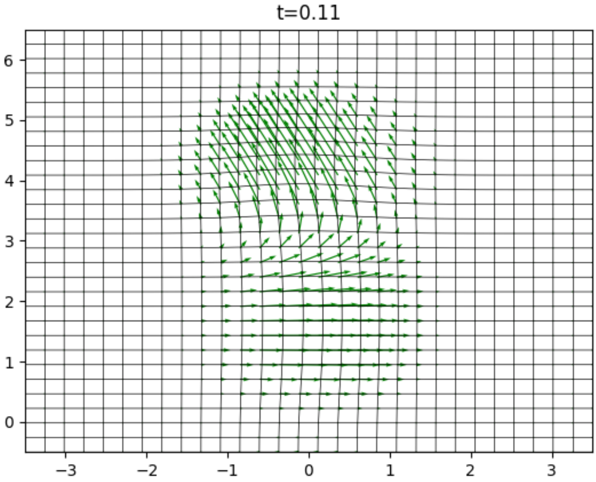

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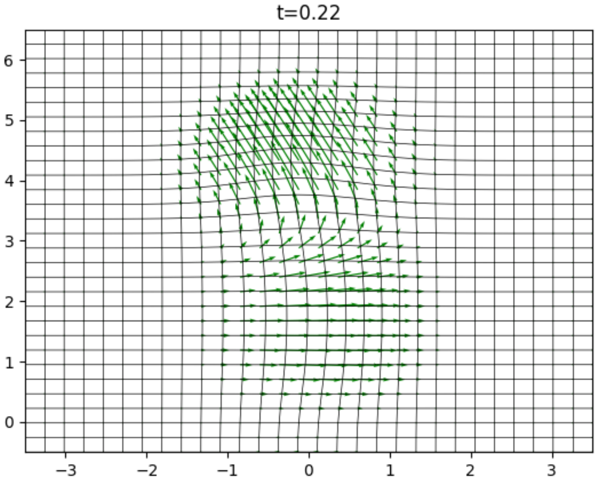

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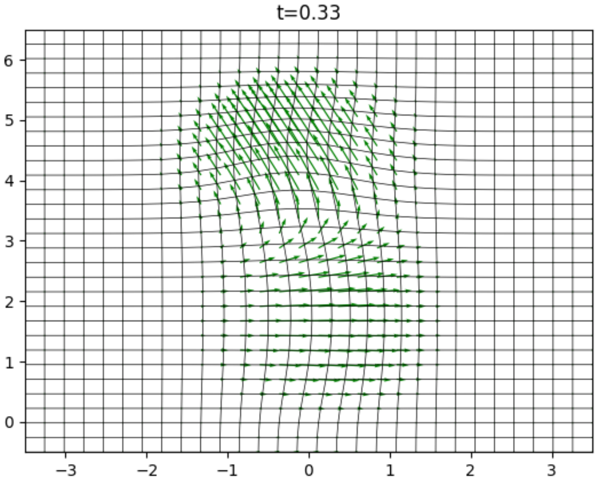

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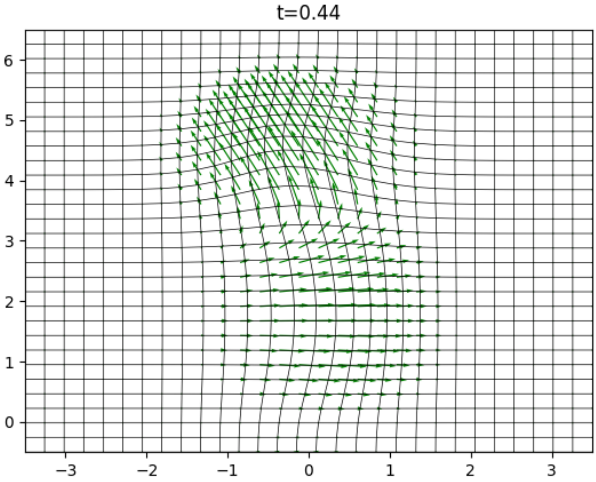

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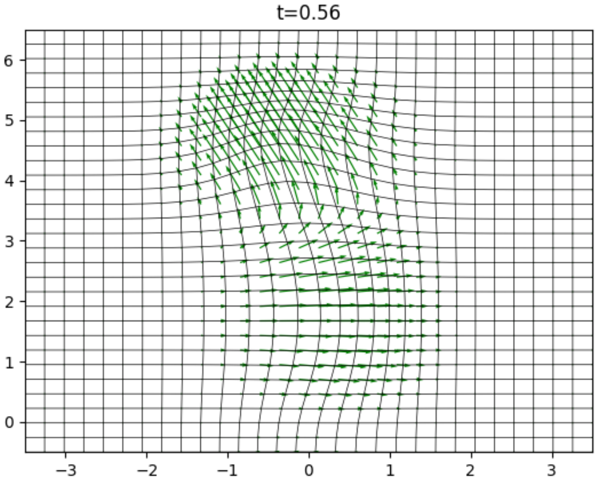

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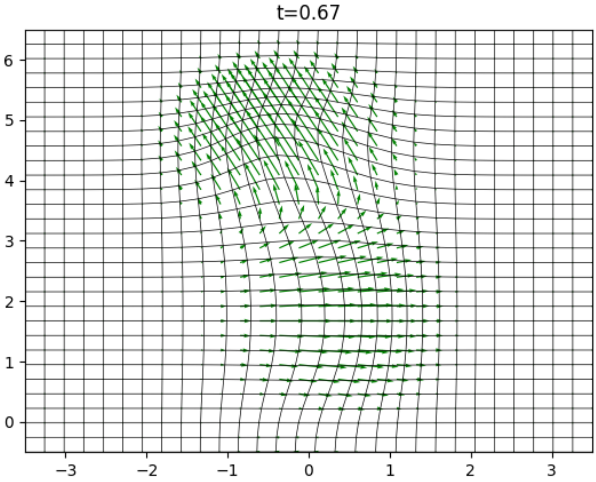

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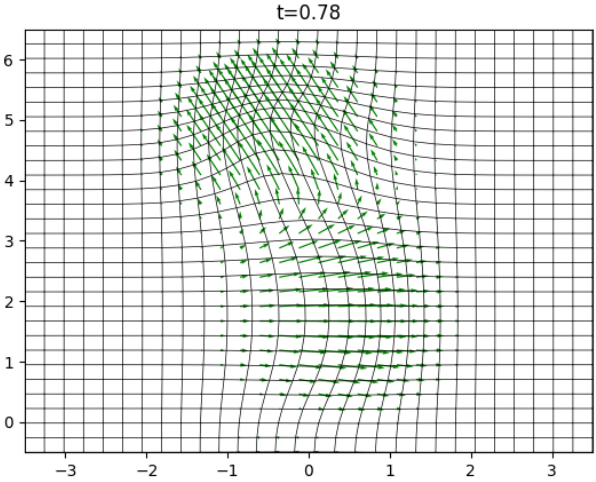

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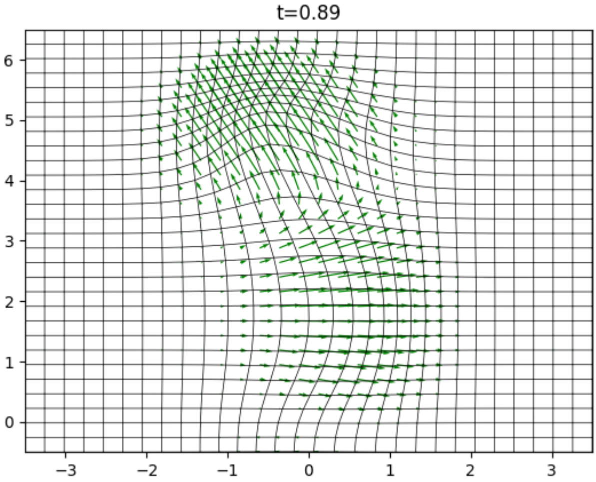

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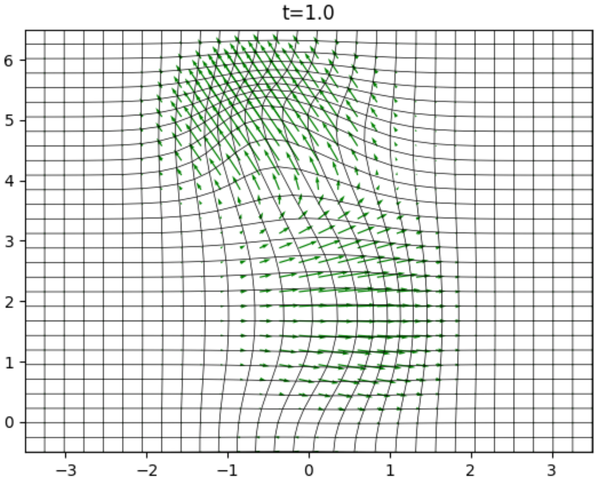


Shape dynamic

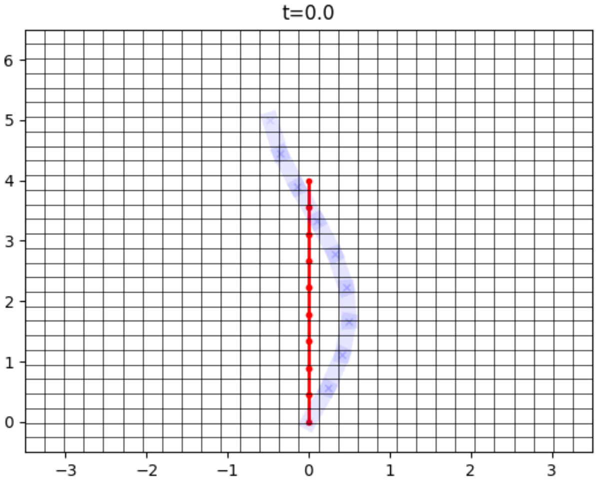
The deformed source is defined by the action of a diffeomorphism on the source

$$q_t = \varphi_t \cdot q^{(0)}$$

From the dynamic of the diffeomorphism, we deduce the dynamic of the deformed shape :

$$\begin{cases} \dot{q}_t &= v_t \cdot q_t \\ q_0 &= q^{(0)} \end{cases}$$

Shape deformation



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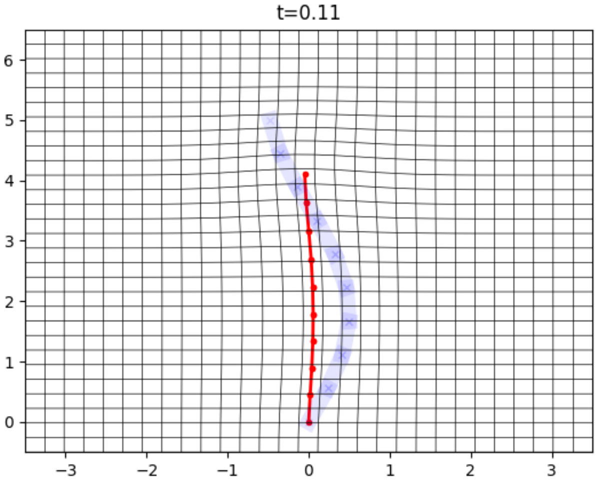

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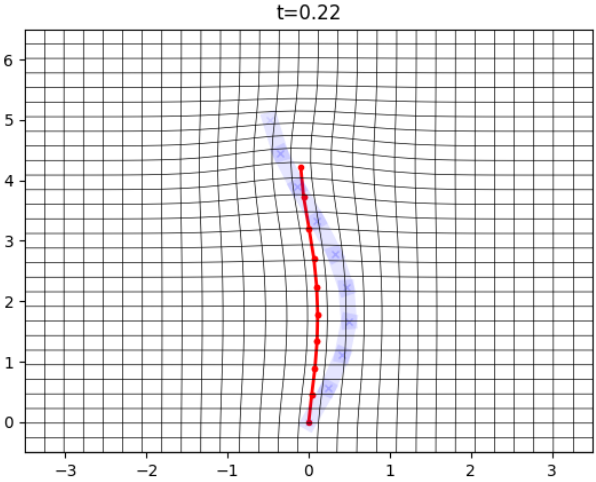
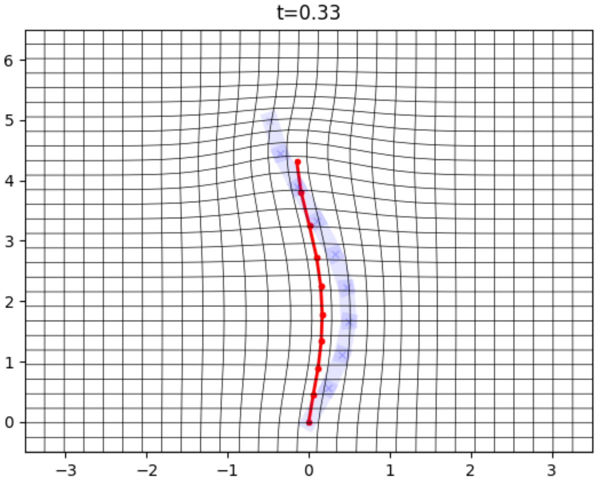


Shape deformation



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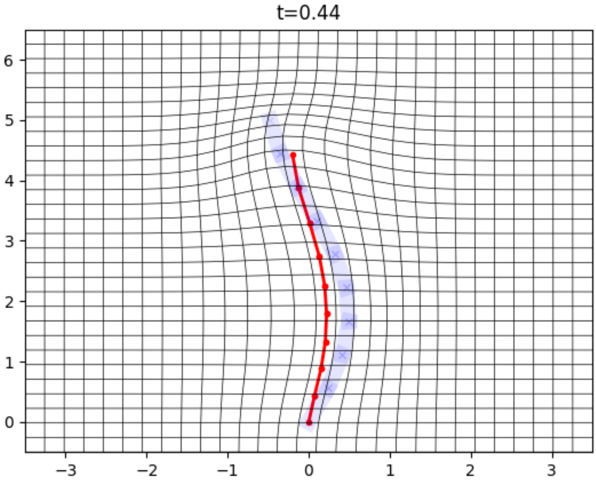

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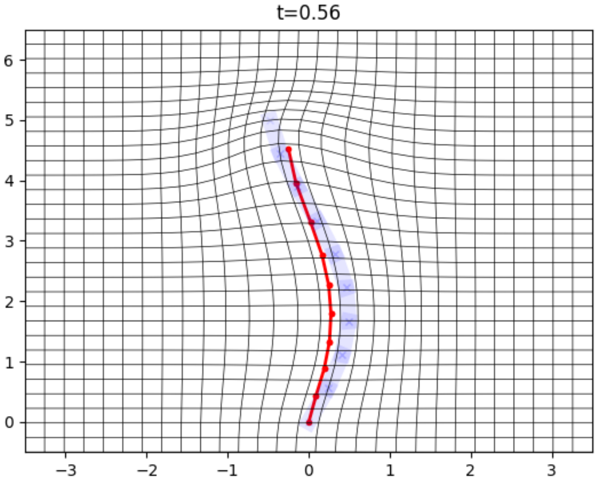

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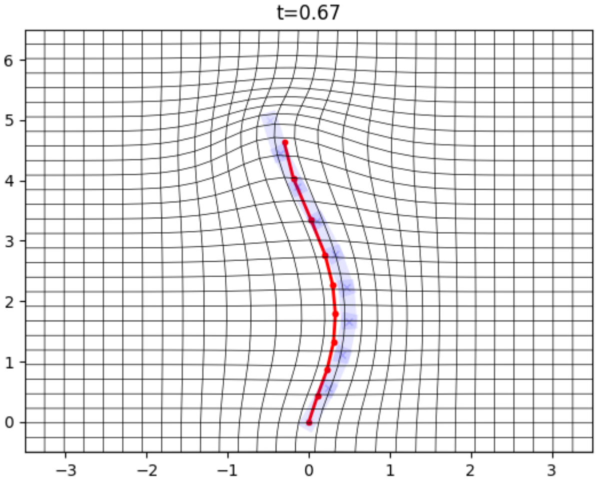

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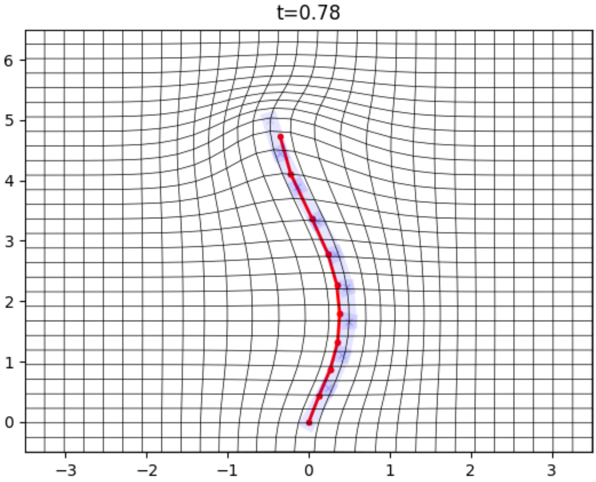

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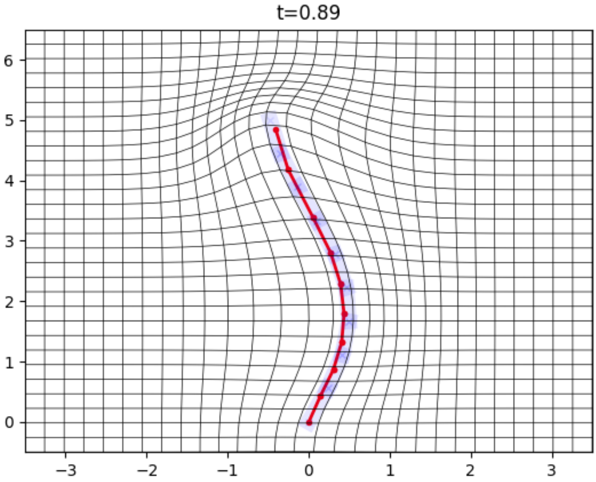

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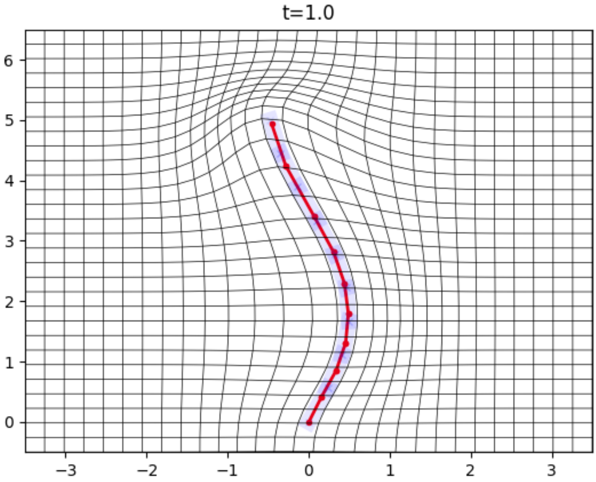
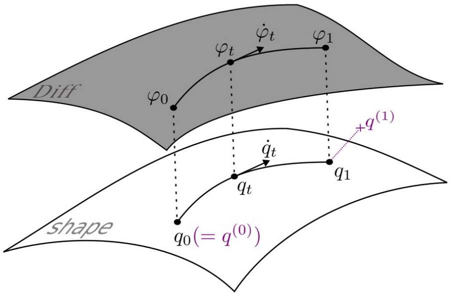


LDDMM : Inexact matching

Shape registration corresponds to the following energy minimization problem :

$$\min_v E(v) = \int_0^1 \frac{1}{2}|v_t|_V^2 \; dt + D(\varphi_1 \cdot q^{(0)}, q^{(1)})$$

$$\text{subject to} \begin{cases} \dot{q}_t & = & v_t \cdot q_t \\ q_0 & = & q^{(0)} \end{cases}$$



Coupling two types of deformations

Let $v \in L^2([0,1], V), w \in L^2([0,1], W)$ be two vector fields and ψ its associated diffeomorphism.

$$\dot{\psi}_t = (v_t + w_t) \cdot \psi_t \qquad \text{s.t} \quad \psi_0 = \mathrm{id}$$

Given a source $q^{(0)}$, the deformed shape $q_t = \psi_t(q^{(0)})$ follows the dynamic

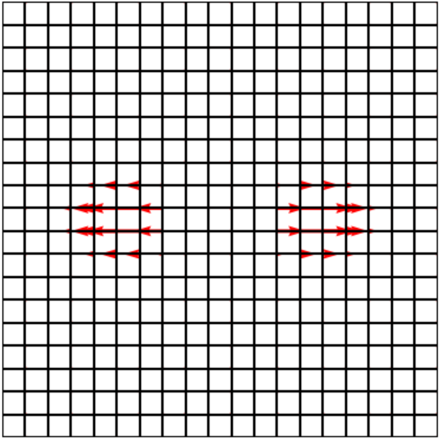
$$\dot{q}_t = v_t \cdot q_t + w_t \cdot q_t$$

The energy minimization problem associated is

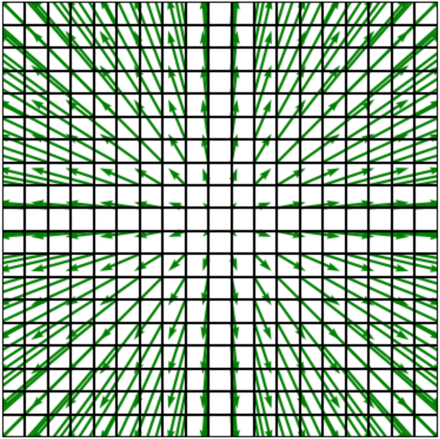
$$\min_{v,w \in L^2([0,1], V \times W)} E(v,w) = \int_0^1 \frac{1}{2}|v_t|_V^2 + \frac{1}{2}|w_t|_W^2 dt + \mathcal{A}(q_1)$$

where $\mathcal{A} : Q \to \mathbb{R}$ is a data attachment term

Decorrelation with respect to a shape

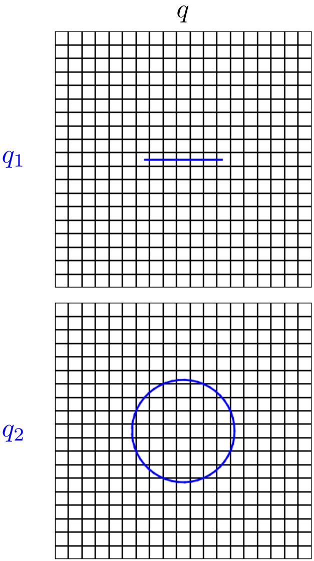


v_1

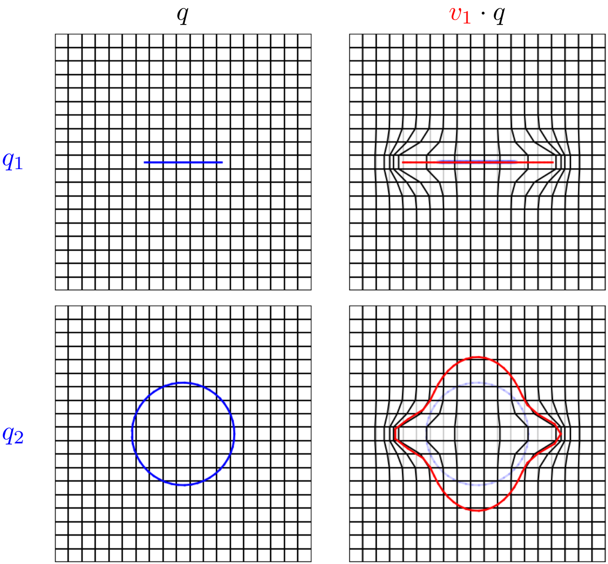


v_2

Decorrelation with respect to a shape



Decorrelation with respect to a shape



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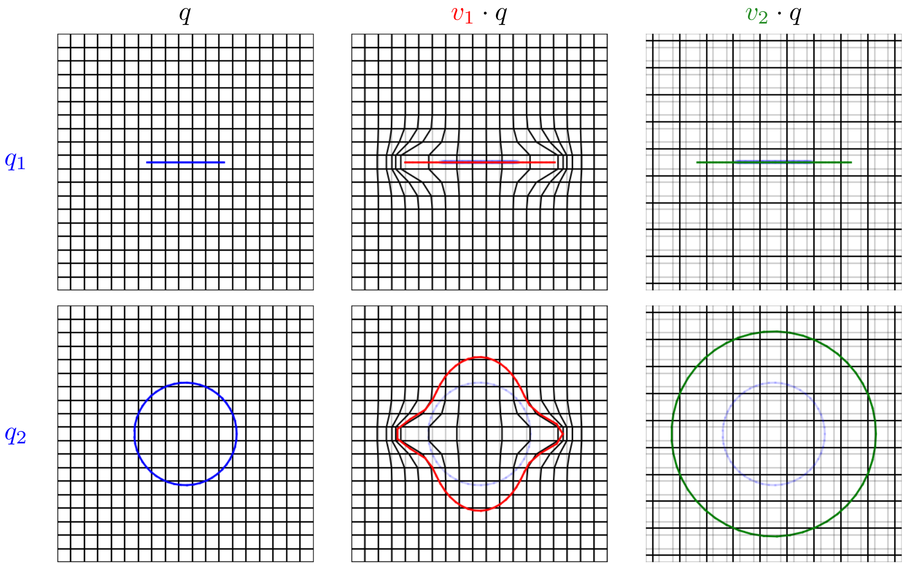


Correlation with respect to a shape

We define the correlation with respect to a shape q between a vector field $v \in V$ and a space of vector fields W by

$$\mathrm{Corr}_q(v, W) = \|w^*\|_W$$

where

$$w^* = \underset{w \in W}{\operatorname{argmin}} \|\delta\mu_q(v) - \delta\mu_q(w)\|_{\mathcal{W}}^2 + \lambda \|w\|_W^2$$

and $\mathcal{W} \hookrightarrow C_0^1(\mathbb{R}^d, \mathbb{R}^d)$ is a Reproducing Kernel Hilbert Space

Varifold

Definition

A varifold is a continuous linear form on $\Omega = \{\omega : \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}\}$.
The varifold μ_q associated to the shape $q : X \to \mathbb{R}^d$ is defined by :

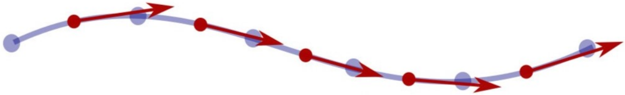
$$\mu_q(\omega) = \int_X \omega(x, \vec{t}(x))\, dx$$

where $\vec{t}$ represents a tangent/normal vector to the curve/surface.

A discrete curve can be modeled by a varifold

$$\mu_q(\omega) = \sum_{(f^1,f^2)\in F} \|q_{f^2} - q_{f^1}\|\ \omega(c(q_f), \vec{t}(q_f))$$

where $c(q_f) = \frac{q_{f1}+q_{f2}}{2}$ and $\vec{t}(q_f) = \frac{q_{f2}-q_{f1}}{\|q_{f2}-q_{f1}\|}$.



Properties

Proposition

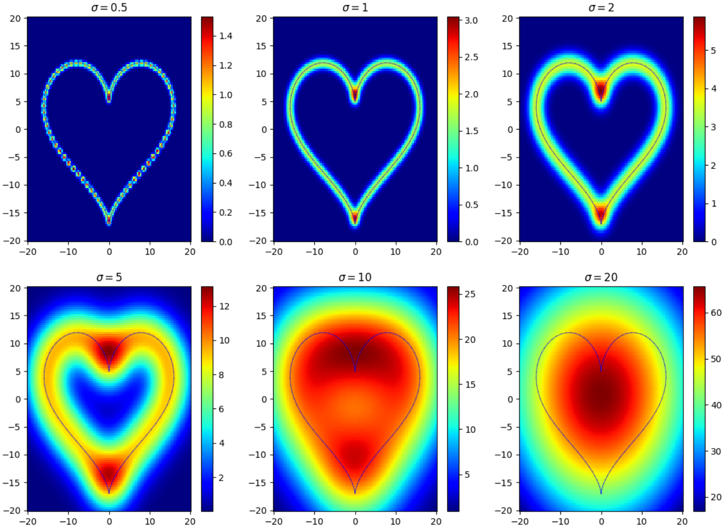
Given a RKHS $\mathcal{W} \hookrightarrow C_0^0(\mathbb{R}^d \times \mathbb{S}^{d-1})$ generated by a kernel $k_{\mathcal{W}} = k_E \otimes k_T$ and two curves q_a and q_b represented by $\mu_{q_a}, \mu_{q_b} \in W'$, there exists a scalar product $\langle \mu_{q_a}, \mu_{q_b} \rangle$.

Proposition

The action of a diffeomorphism on a varifold is defined by

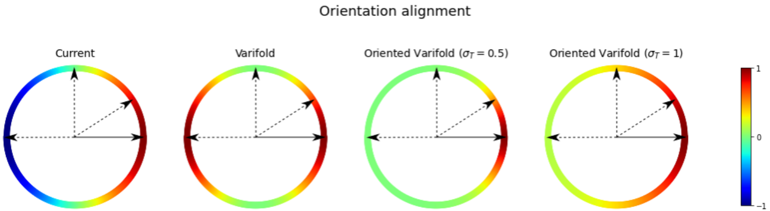
$$(\phi_* \mu_q)(\omega) = \mu_{\phi(q)}(\omega) = \sum_{(f^1, f^2) \in F} \|\phi(q_{f^2}) - \phi(q_{f^1})\| \; \omega\left(c(\phi(q_f)), \vec{t}(\phi(q_f))\right)$$

Gaussian kernel on the positions



Kernels on the tangent

Data type	$\gamma(\langle \overrightarrow{t_x}, \overrightarrow{t_y} \rangle)$	$\gamma(u)$
Currents	$\langle \overrightarrow{t_x}, \overrightarrow{t_y} \rangle$	u
(unoriented) Varifolds	$\langle \overrightarrow{t_x}, \overrightarrow{t_y} \rangle^2$	u^2
Oriented Varifolds	$\exp\left(\frac{-\lVert \overrightarrow{t_x} - \overrightarrow{t_y} \rVert^2}{\sigma_T^2}\right)$	$\exp\left(\frac{2u-2}{\sigma_T^2}\right)$



Speed of a varifold induced by a vector field

Theorem (Charon, Trouvé 2013)

Let $t \mapsto \phi_t$ *be a flow of diffeomorphism such that* $\phi_0 = \mathrm{id}$ *and* $\dot{\phi}_t|_{t=0} = v$

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \mu_{\phi_t(q)}(\omega) &= \sum_{(f^1,f^2)\in F} \frac{\langle v(q_{f^2}) - v(q_{f^1}), q_{f^1} - q_{f^2}\rangle}{\|q_{f^1} - q_{f^2}\|}\, \omega(c(q_f), \vec{t}(q_f)) \\ &+ \|q_{f^1} - q_{f^2}\| \left(\partial_x \omega(c(q_f), \vec{t}(q_f)) \Big| v(c(q_f))\right) \\ &+ \|q_{f^1} - q_{f^2}\| \left(\partial_{\vec{t}} \omega(c(q_f), \vec{t}(q_f)) \Big| \nabla^{\perp} v(q_f)\right)\end{aligned}$$

where $\nabla^{\perp} v(q_f) = \frac{v(q_{f^2}) - v(q_{f^1})}{\|q_{f^1} - q_{f^2}\|} - \left(\frac{v(q_{f^2}) - v(q_{f^1})}{\|q_{f^1} - q_{f^2}\|} \cdot \vec{t}(q_f)\right) \vec{t}(q_f)$

In the following, we will denote $\frac{d}{dt}\Big|_{t=0} \mu_{\phi_t(q)}$ as $\delta\mu_q(v)$

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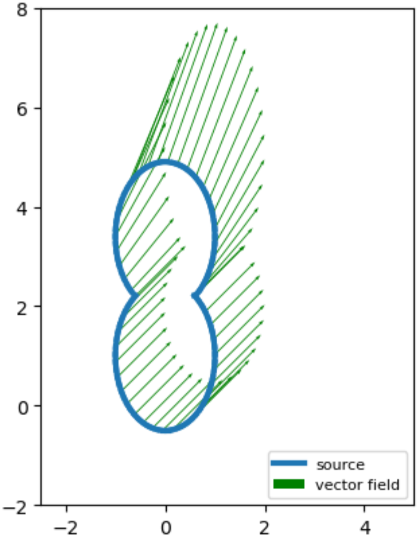
Influence of σ : $k(r) = e^{-r^2/\sigma^2}$



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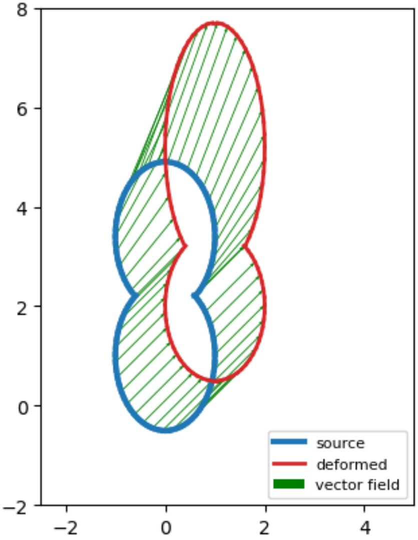

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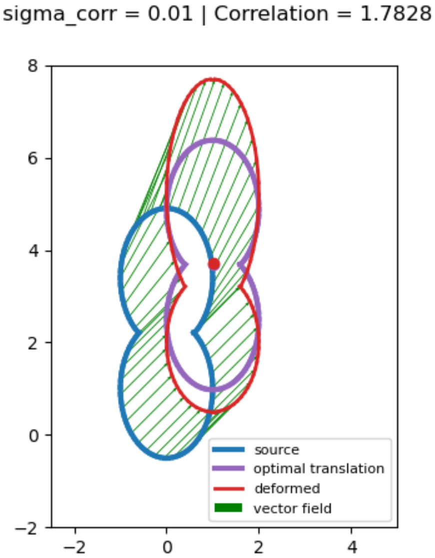

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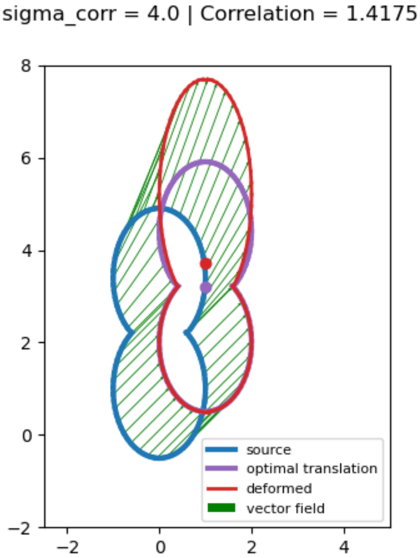

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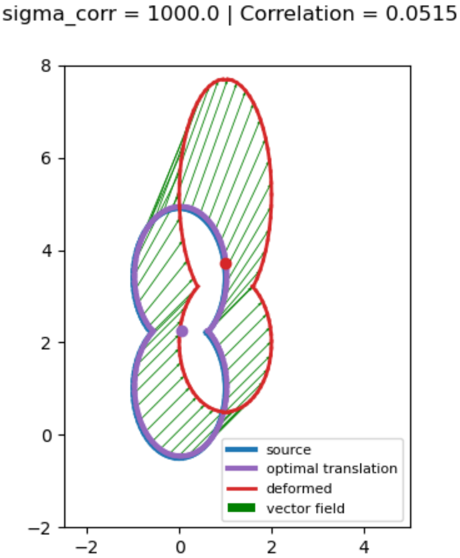

Dynamic generated by two vector fields

Given V, W two spaces of vector fields, we are interested in the following energy-minimization problem

$$\min_{(v,w)\in L^2([0,1],V\times W)} E(v,w) = \int_0^1 \frac{1}{2}|v_t|_V^2 + \frac{1}{2}|w_t|_W^2 dt + \mathcal{A}(q_1)$$
$$\text{s.t} \quad \dot{q}_t = v_t \cdot q_t + w_t \cdot q_t$$

where $\mathcal{A} : \mathcal{Q} \to \mathbb{R}$ is a data attachment term.

Different approaches in the litterature :

- Multiscale kernel bundle, sum of gaussian kernel : Sommer et al. 2013, Risser 2011
- Semidirect product : Bruveris et al. (2010, 2012)
- Hierarchical model : Pierron et al. 2024

Direct model

We consider the flow of diffeomorphism ψ generated by $(v, w) \in L^2([0,1], V \times W)$

$$\dot{\psi}_t = (v_t + w_t) \circ \psi_t \quad \text{where} \quad \psi_0 = \mathrm{id}$$

Considering the deformed shape $q_t = \psi_t(q^{(0)})$ the energy minimization problem is equivalent to

$$\min_{p_0} E(p_0) = \int_0^1 \frac{1}{2}|v_t|_V^2 + \frac{1}{2}|w_t|_W^2 dt + \mathcal{A}(q_1)$$

$$\text{s.t} \quad \begin{cases} \dot{q}_t & = \xi_{q_t}^V(v_t) + \xi_{q_t}^W(w_t) \\ \dot{p}_t & = -(\partial_q \xi_{q_t}^V(v_t) + \partial_q \xi_{q_t}^W(w_t))^* p_t \\ v_t & = K_V \xi_{q_t}^{V*} p_t \\ w_t & = K_W \xi_{q_t}^{W*} p_t \end{cases}$$

where $\xi_{q_t}^V(v_t) = v_t \cdot q_t$, $\xi_{q_t}^W(w_t) = w_t \cdot q_t$ and $p_t \in T_{q_t}^* Q$

Direct model

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where $\xi_{q_t}^V(v_t) = v_t \cdot q_t$, $\xi_{q_t}^W(w_t) = w_t \cdot q_t$ and $p_t \in T_{q_t}^* Q$

Direct model

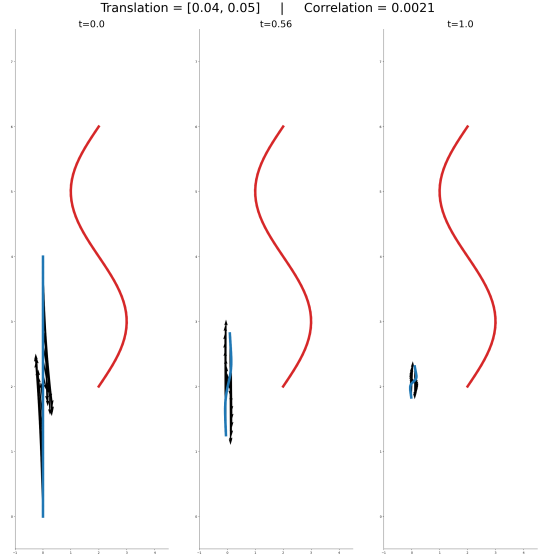
We penalize the energy with the correlation to define a new problem.

$$\min_{p_0} E(p_0) = \int_0^1 \frac{1}{2}|v_t|_V^2 + \frac{1}{2}|w_t|_W^2 dt + \gamma \int_0^1 \frac{1}{2} \mathrm{Corr}_{q_t}(v_t, W)^2 dt + \mathcal{A}(q_1)$$

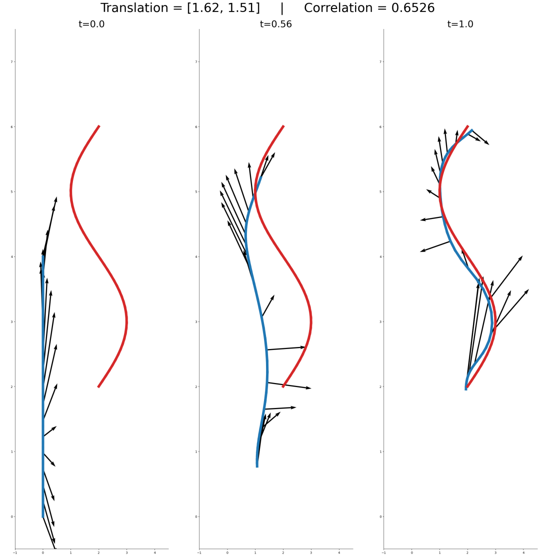
$$\text{s.t} \quad \begin{cases} \dot{q}_t &= \xi_{q_t}^V(v_t) + \xi_{q_t}^W(w_t) \\ \dot{p}_t &= -(\partial_q \xi_{q_t}^V(v_t) + \partial_q \xi_{q_t}^W(w_t))^* p_t \\ v_t &= K_V \xi_{q_t}^{V*} p_t \\ w_t &= K_W \xi_{q_t}^{W*} p_t \end{cases}$$

where $\xi_{q_t}^V(v_t) = v_t \cdot q_t$, $\xi_{q_t}^W(w_t) = w_t \cdot q_t$ and $p_t \in T_{q_t}^* Q$.

Decorrelation with one moment



Decorrelation with one moment



Direct model

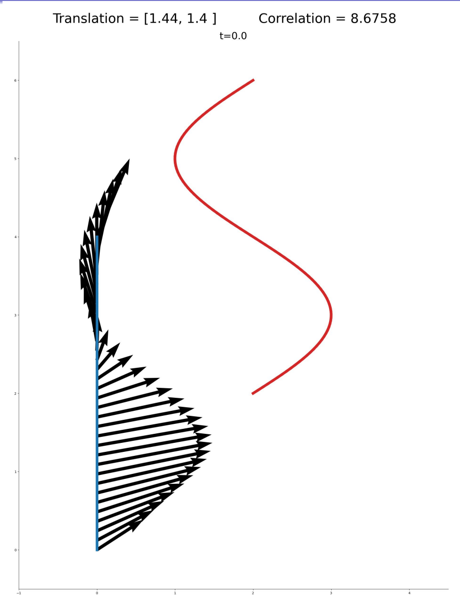
Considering the partial gradient of $E(v, w)$, we can define a new problem parameterized by two moments $(p_t^V, p_t^W) \in T^*_{q_t}Q \times T^*_{q_t}Q$.

$$\min_{p_0^V, p_0^W} E(p_0^V, p_0^W) = \int_0^1 \frac{1}{2}|v_t|_V^2 + \frac{1}{2}|w_t|_W^2 dt + \lambda \int_0^1 \frac{1}{2}\operatorname{Corr}_{q_t}(v_t, W)^2 dt + \mathcal{A}(q_1)$$

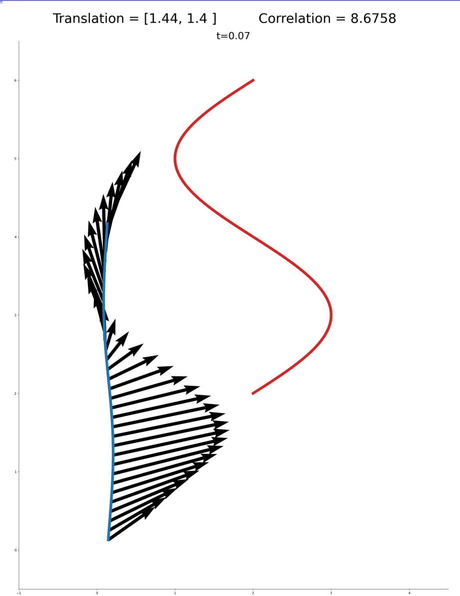
$$\text{s.t} \quad \begin{cases} \dot{q}_t &= v_t \cdot q_t + w_t \cdot q_t \\ \dot{p}_t^V &= -(\partial_q \xi_{q_t}^V(v_t) + \partial_q \xi_{q_t}^W(w_t))^* p_t^V \\ \dot{p}_t^W &= -(\partial_q \xi_{q_t}^V(v_t) + \partial_q \xi_{q_t}^W(w_t))^* p_t^W \\ v_t &= K_V \xi_{q_t}^{V*} p_t^V \\ w_t &= K_W \xi_{q_t}^{W*} p_t^W \end{cases}$$

where $\xi_{q_t}^V(v_t) = v_t \cdot q_t$, $\xi_{q_t}^W(w_t) = w_t \cdot q_t$.

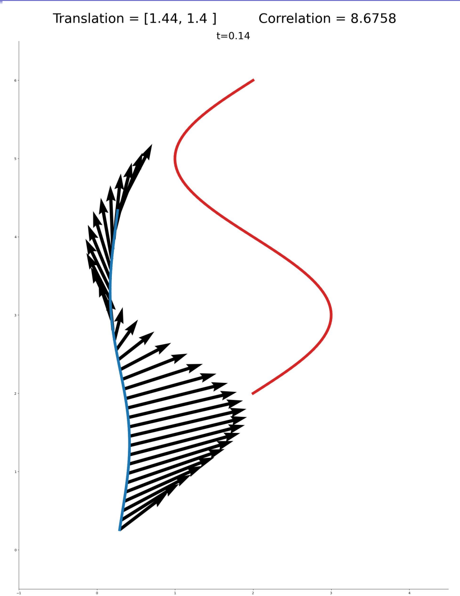
Direct model without decorrelation



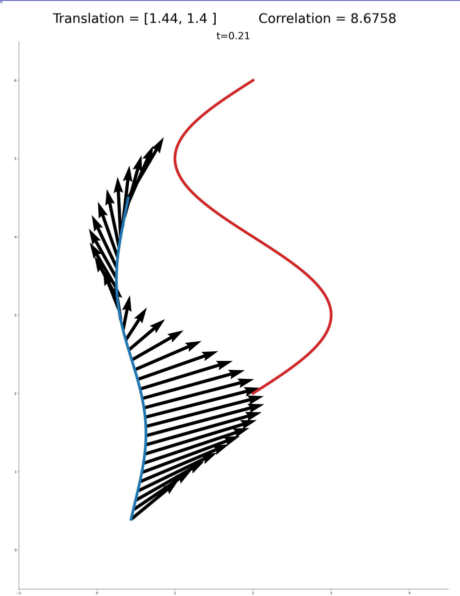
Direct model without decorrelation



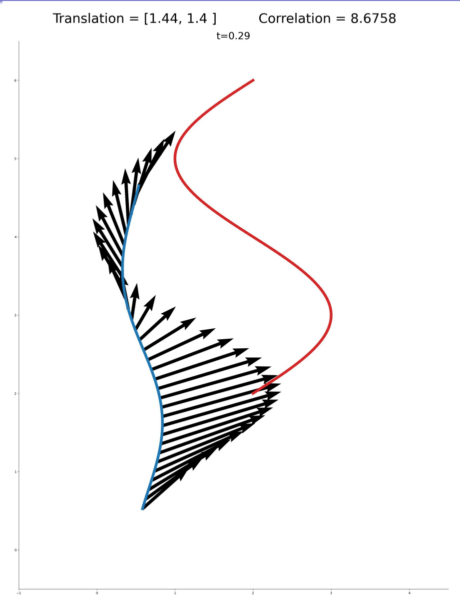
Direct model without decorrelation



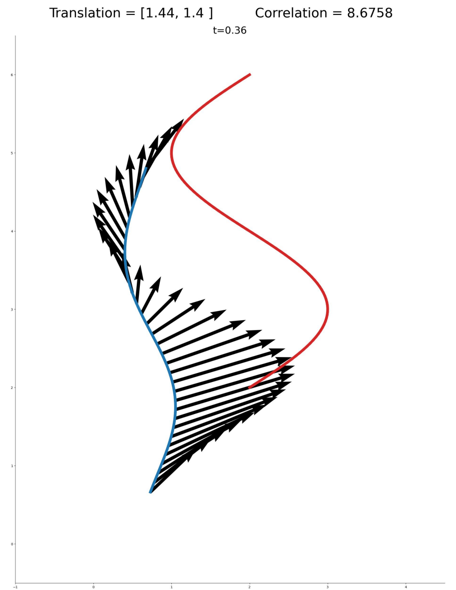
Direct model without decorrelation



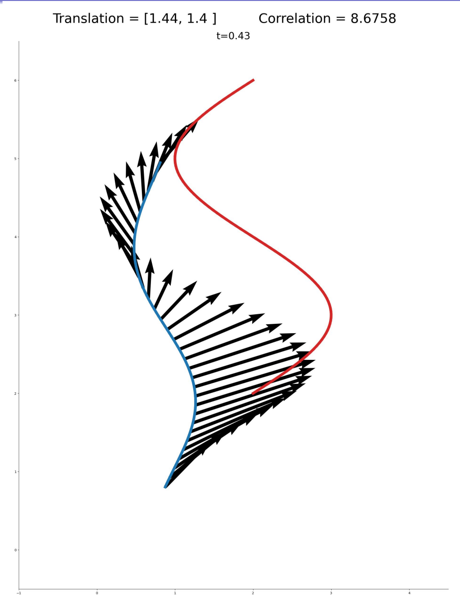
Direct model without decorrelation



Direct model without decorrelation



Direct model without decorrelation



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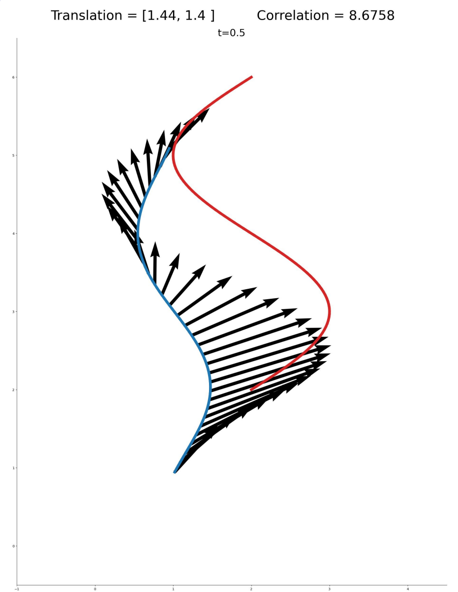
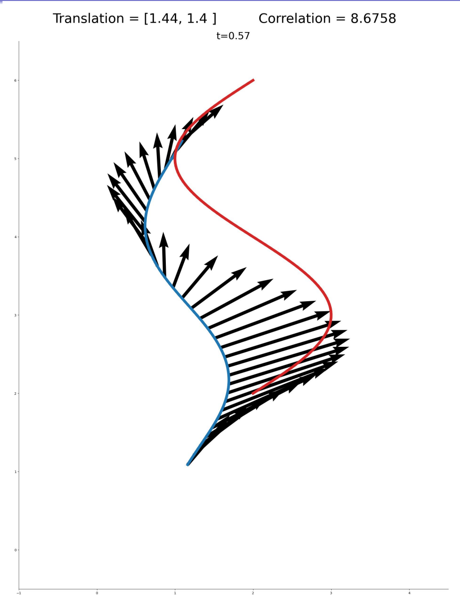
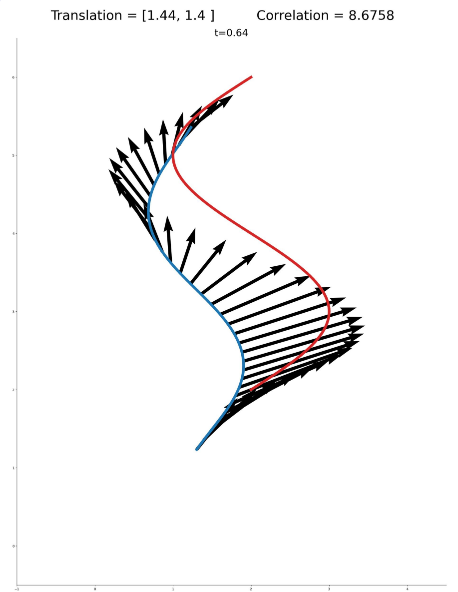


Direct model without decorrelation

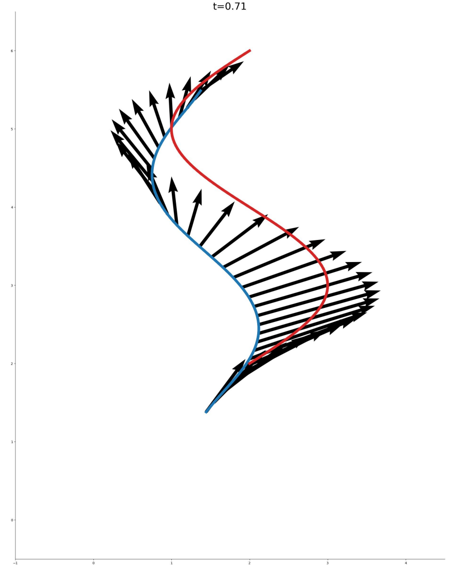


Direct model without decorrelation

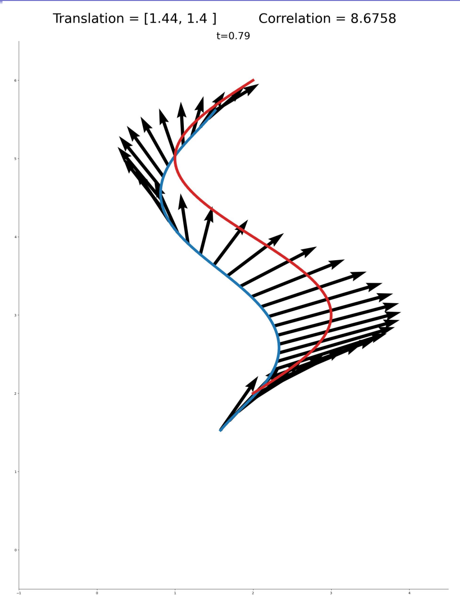


Direct model without decorrelation

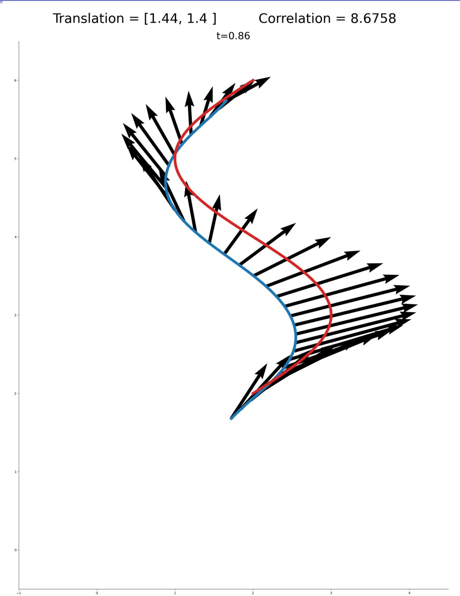
Translation = [1.44, 1.4]   Correlation = 8.6758



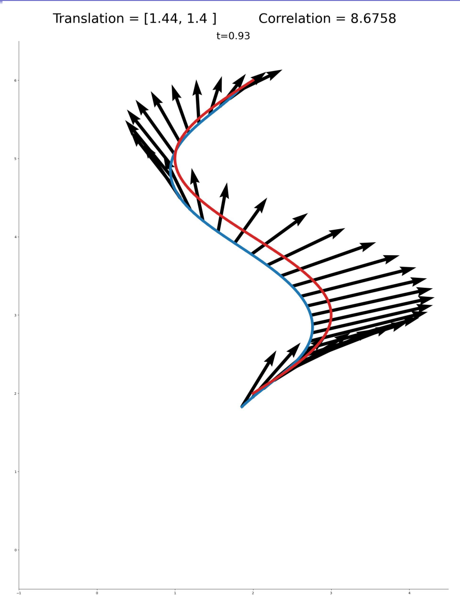
Direct model without decorrelation



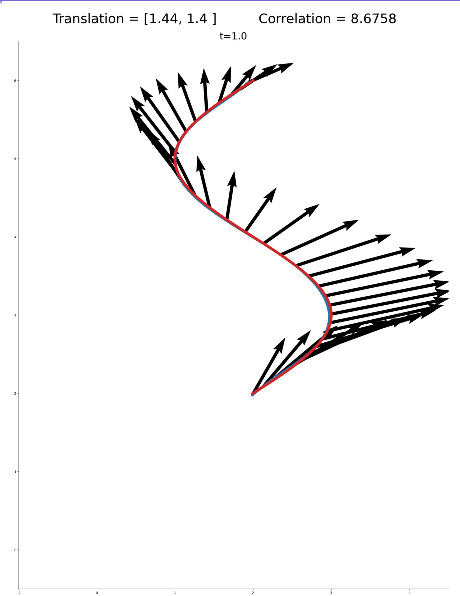
Direct model without decorrelation



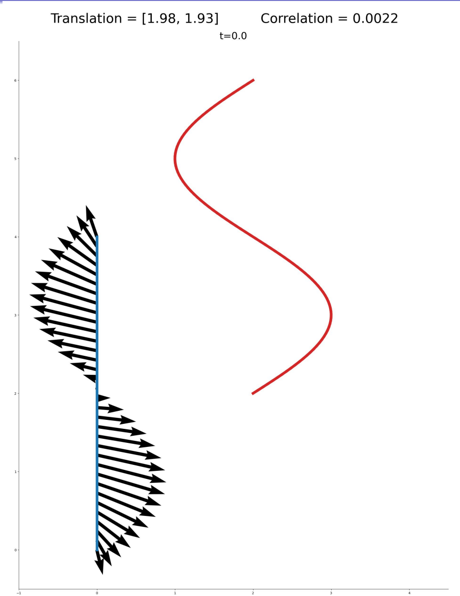
Direct model without decorrelation



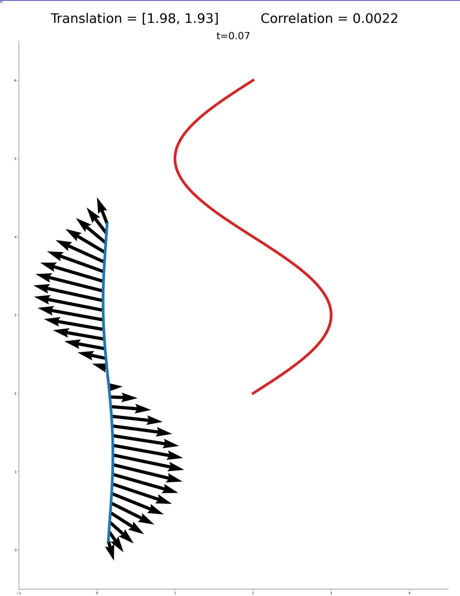
Direct model without decorrelation



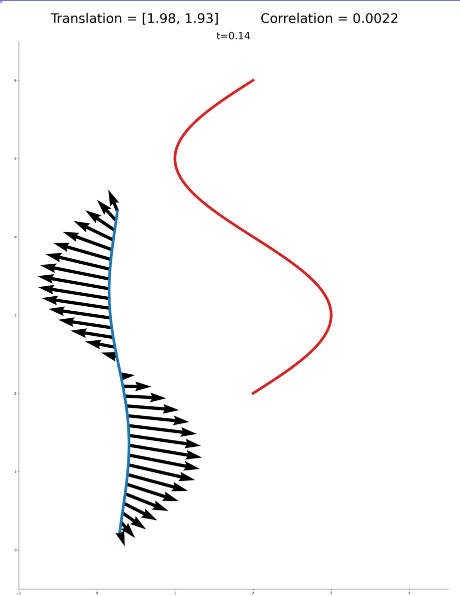
Direct model with decorrelation



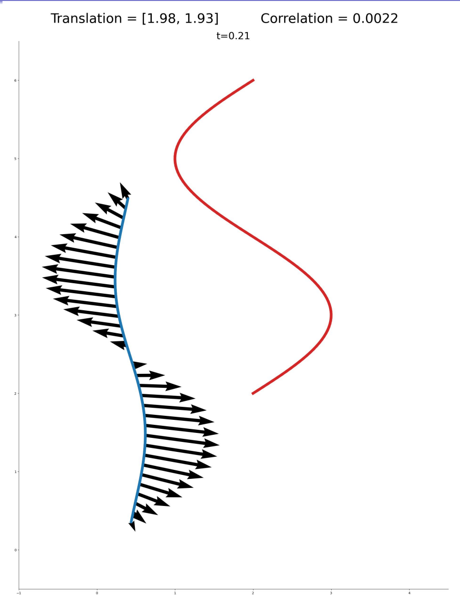
Direct model with decorrelation



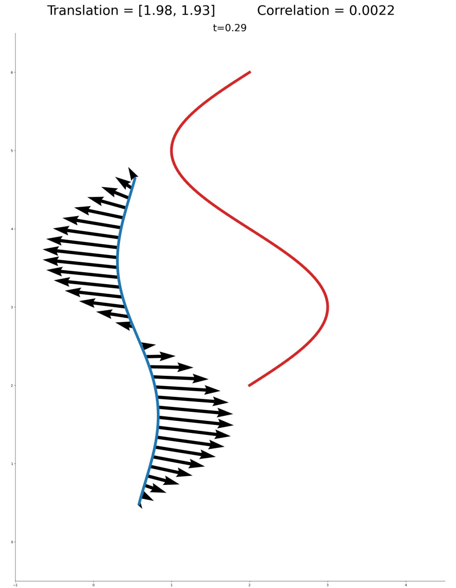
Direct model with decorrelation



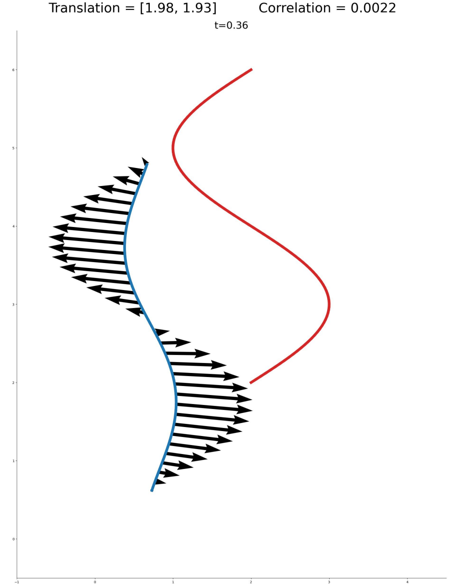
Direct model with decorrelation



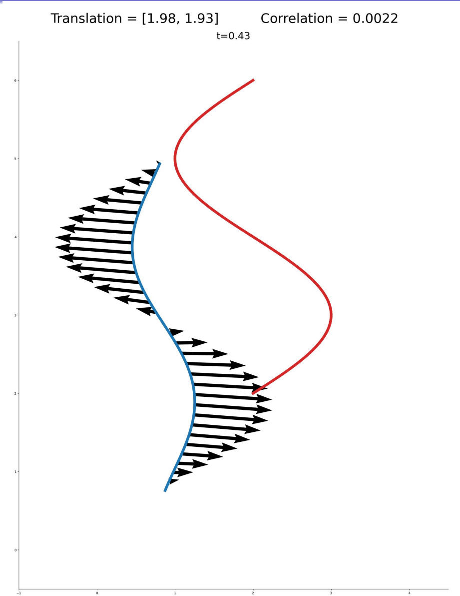
Direct model with decorrelation



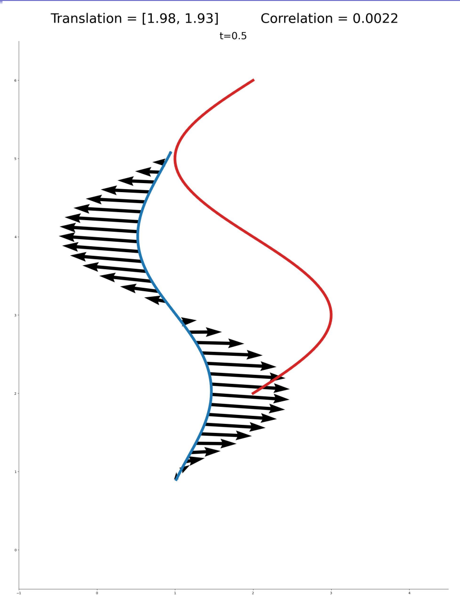
Direct model with decorrelation



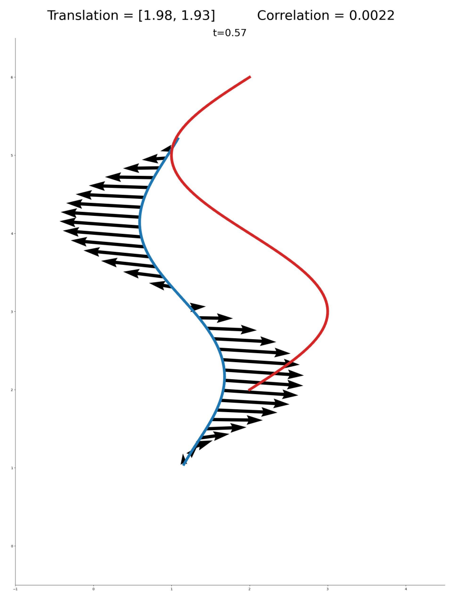
Direct model with decorrelation



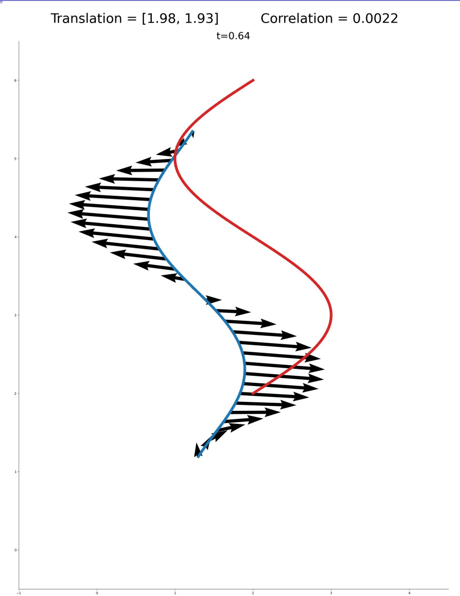
Direct model with decorrelation



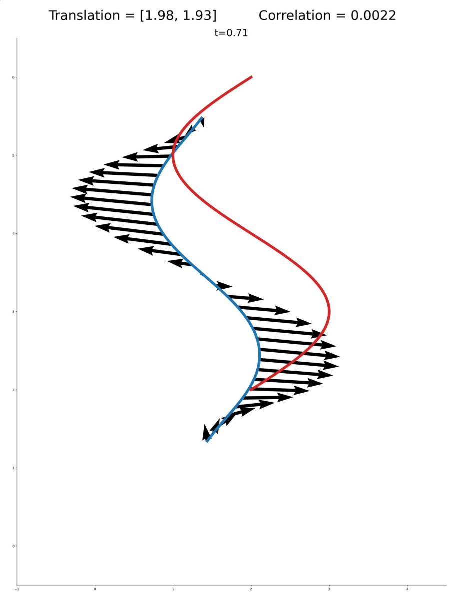
Direct model with decorrelation



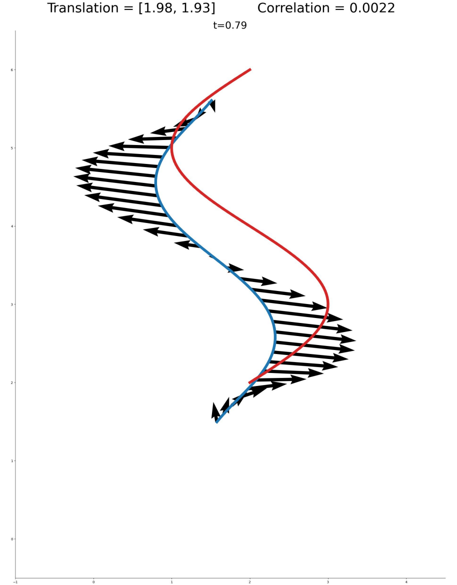
Direct model with decorrelation



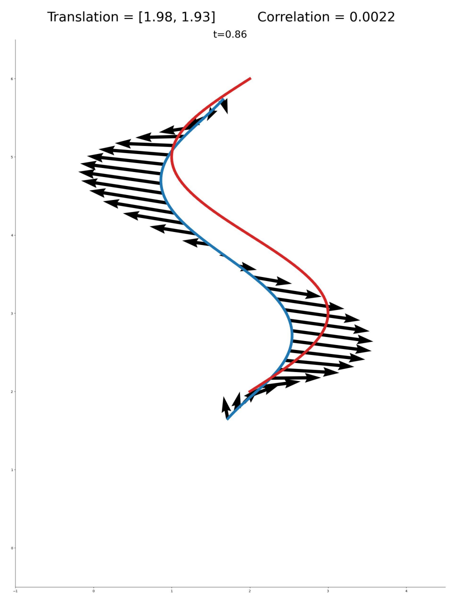
Direct model with decorrelation



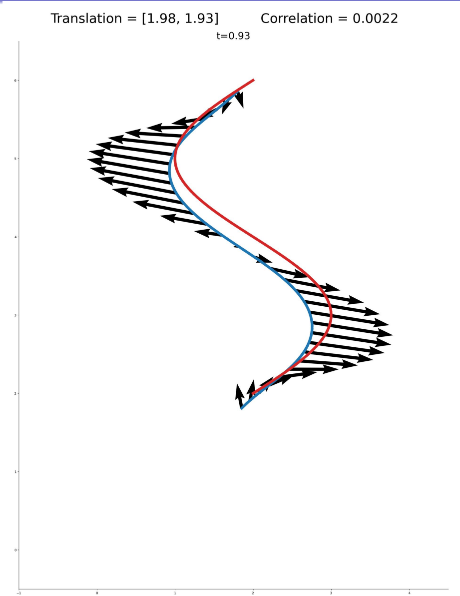
Direct model with decorrelation



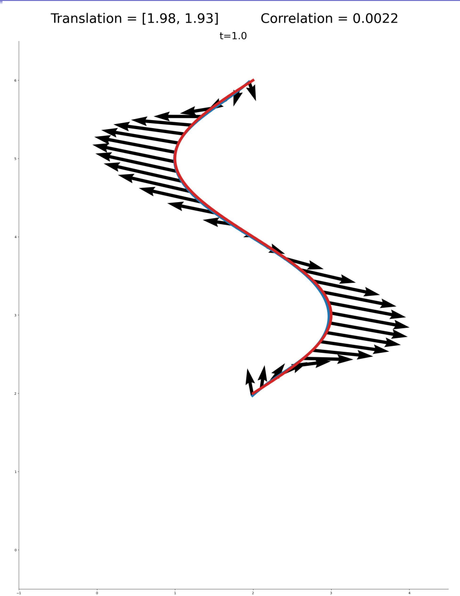
Direct model with decorrelation



Direct model with decorrelation



Direct model with decorrelation



Semidirect model (joint work with Thomas Pierron)

Let G be a finite dimensional Lie group and $\mathfrak{g}$ its Lie algebra. We denote $\alpha_g(\varphi)$ the action of G on $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$.

We consider the semidirect product $\mathcal{G} = G \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ and we assume it acts on Q as follow :

$$(g, \varphi) \cdot q = g \cdot (\varphi \cdot q)$$

Example : If $G = \mathbb{R}^d$, then $\alpha_T(\varphi)(x) = \varphi(x + T) - T$ and $(T, \varphi) \cdot q = \varphi(q) + T$.

Semidirect model

The minimization problem associated to the semidirect model is

$$\min_{p_0} E(p_0) = \int_0^1 \frac{1}{2}|v_t|_V^2 + \frac{1}{2}|X_t|_{\mathfrak{g}}^2 dt + \mathcal{A}(q_1)$$

$$\text{s.t} \quad \begin{cases} \dot{q}_t & = v_t \cdot q_t + X_t \cdot q_t \\ \dot{p}_t & = -(\partial_q \xi_{q_t}^V(v_t) + \partial_q \xi_{q_t}^{\mathfrak{g}}(X_t))^* p_t \\ v_t & = K_V \xi_{q_t}^{V*} p_t \\ X_t & = K_{\mathfrak{g}} \xi_{q_t}^{\mathfrak{g}*} p_t \end{cases}$$

where $\xi_{q_t}^{\mathfrak{g}}(X_t) = X_t \cdot q_t$ and $p_t \in T_{q_t}^* Q$

Semidirect model

The minimization problem associated to the semidirect model is

$$\min_{p_0} E(p_0) = \int_0^1 \frac{1}{2}|v_t|_V^2 + \frac{1}{2}|X_t|_{\mathfrak{g}}^2 dt + \mathcal{A}(q_1)$$

$$\text{s.t} \quad \begin{cases} \dot{q}_t & = v_t \cdot q_t + X_t \cdot q_t \\ \dot{p}_t & = -(\partial_q \xi_{q_t}^V(v_t) + \partial_q \xi_{q_t}^{\mathfrak{g}}(X_t))^* p_t \\ v_t & = K_V \xi_{q_t}^{V*} p_t \\ X_t & = K_{\mathfrak{g}} \xi_{q_t}^{\mathfrak{g}*} p_t \end{cases}$$

where $\xi_{q_t}^{\mathfrak{g}}(X_t) = X_t \cdot q_t$ and $p_t \in T_{q_t}^* Q$

Semidirect model

We define a new shape $\tilde{q} = g^{-1} \cdot q$, in particular the deformation of $\tilde{q}$ is

$$\tilde{q}_t = \varphi_t \cdot q^{(0)}$$

Considering the augmented shape space $G \times Q$ and a new data attachment term

$$\tilde{A}(g, \tilde{q}) = \mathcal{A}(g \cdot \tilde{q}) = \mathcal{A}(q)$$

will allow us to consider two moments $p^{\mathfrak{g}} \in T^*G$ and $\tilde{p} \in T^*Q$.

Semidirect model

The minimization energy problem can be written with two moments $(\tilde{p}_t, p_t^{\mathfrak{g}}) \in T^*_{\tilde{q}_t} Q \times T^*_{g_t} G$:

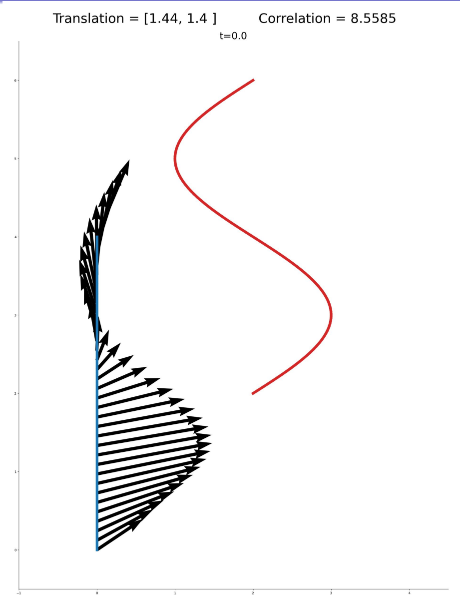
$$\min_{\tilde{p}_0, p_0^{\mathfrak{g}}} E(\tilde{p}_0, p_0^{\mathfrak{g}}) = \int_0^1 \frac{1}{2}|v_t|_V^2 + \frac{1}{2}|X_t|_{\mathfrak{g}}^2 dt + \lambda \int_0^1 \frac{1}{2} \mathrm{Corr}_{q_t}(v_t, \mathfrak{g})^2 dt + \tilde{\mathcal{A}}(g_1, \tilde{q}_1)$$

s.t

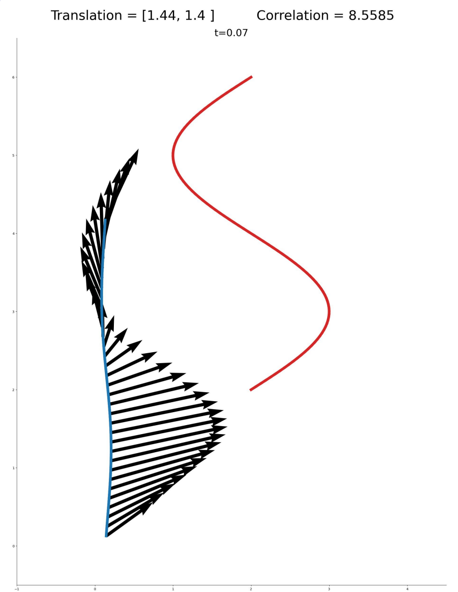
$$\begin{cases} \dot{g}_t &= X_t \cdot g_t \\ \dot{\tilde{q}}_t &= d_{\mathrm{id}}\alpha_{g_t}(v_t) \cdot \tilde{q}_t \\ \dot{\tilde{p}}_t &= -(\partial_q \xi^{\tilde{V}}_{\tilde{q}_t}(d_{\mathrm{id}}\alpha_{g_t}(v_t))^* \tilde{p}_t \\ \dot{p}_t^{\mathfrak{g}} &= -(\partial_g \xi^{\mathfrak{g}}_{g_t}(X_t))^* p_t^{\mathfrak{g}} - (\partial_g \xi^V_{\tilde{q}_t}(d_{\mathrm{id}}\alpha_{g_t}(v_t)))^* \tilde{p}_t \\ v_t &= K_V(\xi^V_{\tilde{q}_t} d_{\mathrm{id}}\alpha_{g_t})^* \tilde{p}_t \\ X_t &= K_{\mathfrak{g}} \xi^{\mathfrak{g}*}_{g_t} p_t^{\mathfrak{g}} \end{cases}$$

where $\xi^{\mathfrak{g}}_{g_t}(X_t) = X_t g_t$

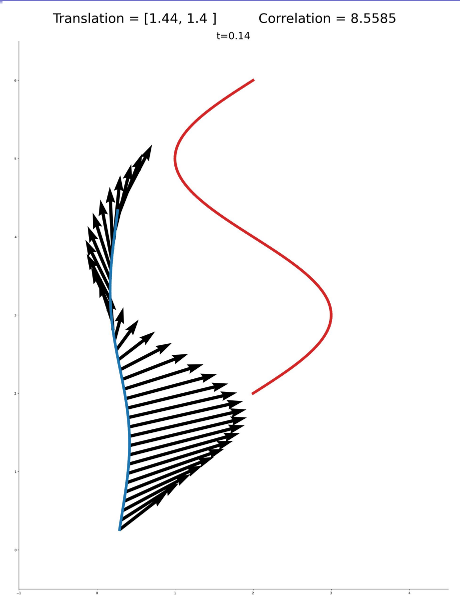
Semidirect model without decorrelation



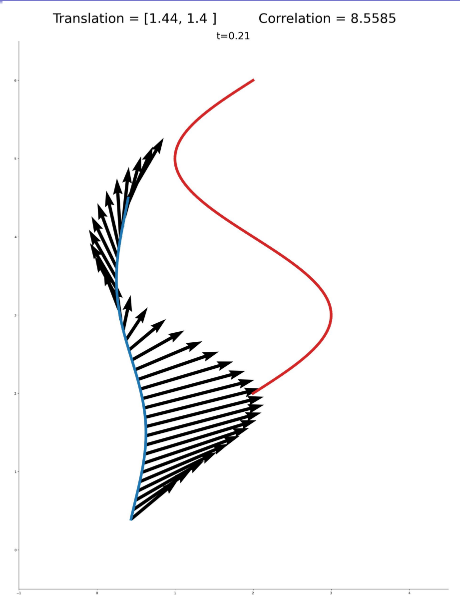
Semidirect model without decorrelation



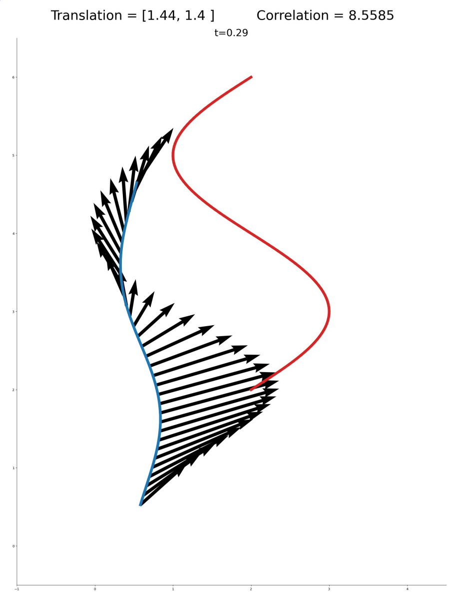
Semidirect model without decorrelation



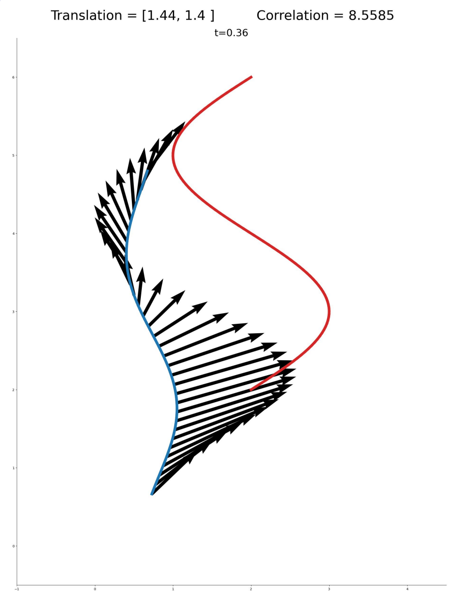
Semidirect model without decorrelation



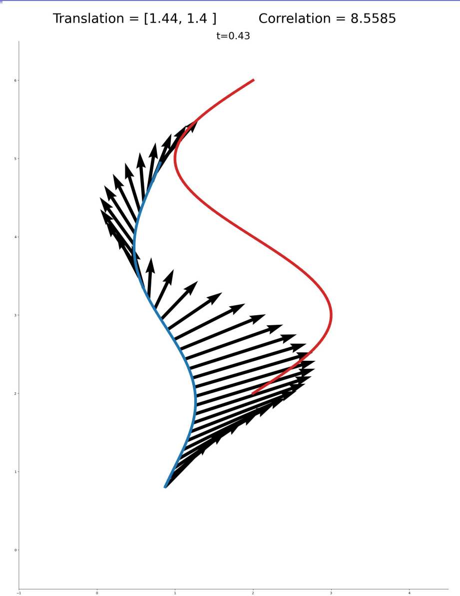
Semidirect model without decorrelation



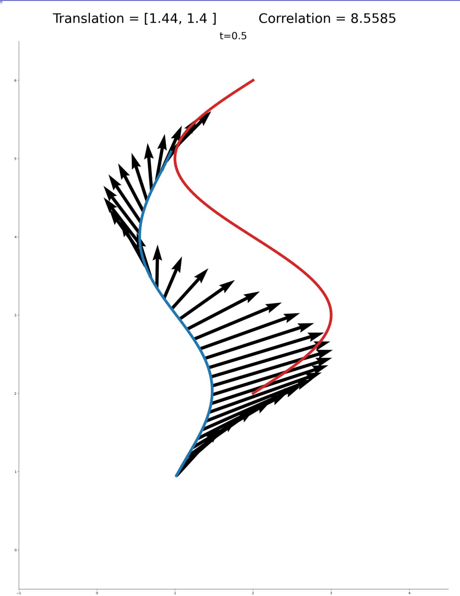
Semidirect model without decorrelation



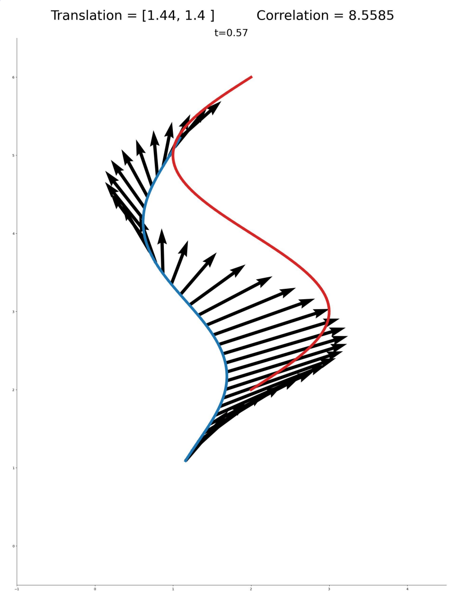
Semidirect model without decorrelation



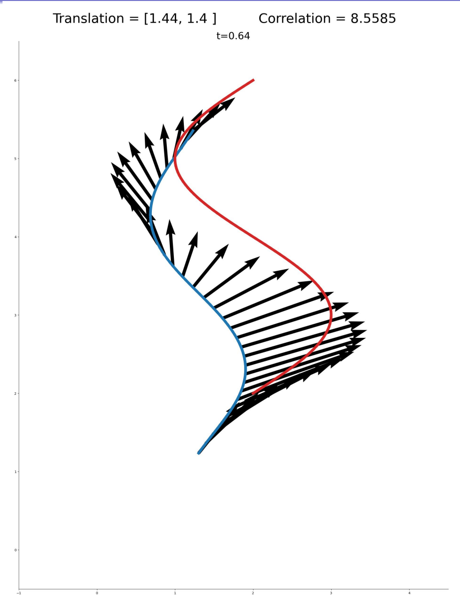
Semidirect model without decorrelation



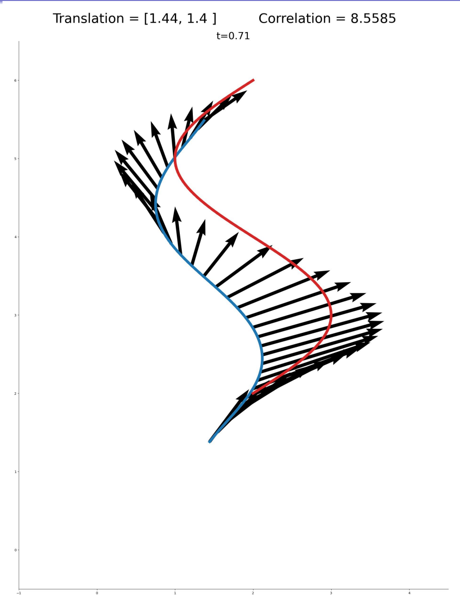
Semidirect model without decorrelation



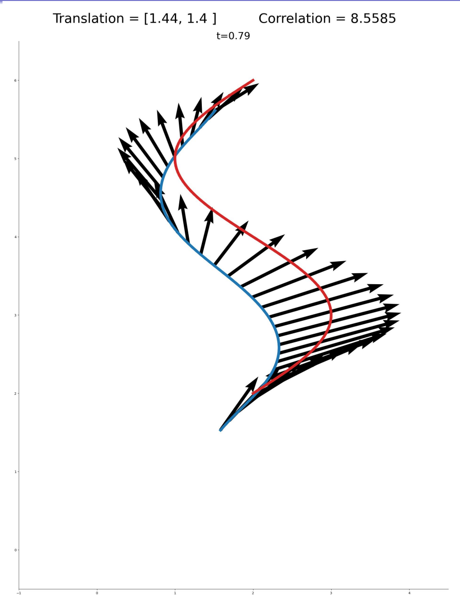
Semidirect model without decorrelation



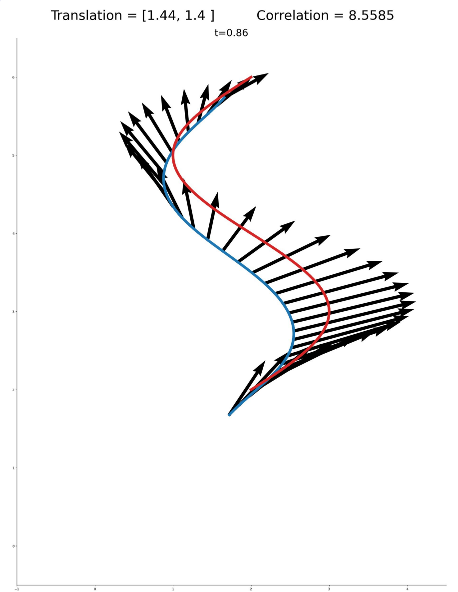
Semidirect model without decorrelation



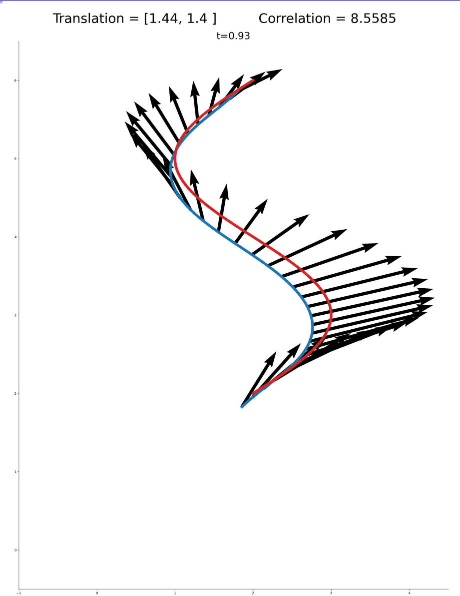
Semidirect model without decorrelation



Semidirect model without decorrelation



Semidirect model without decorrelation



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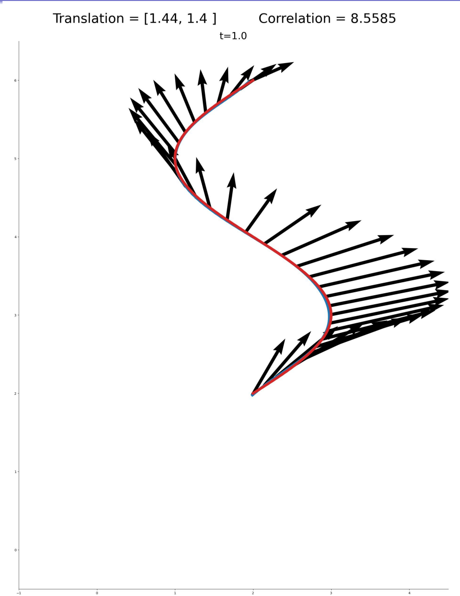

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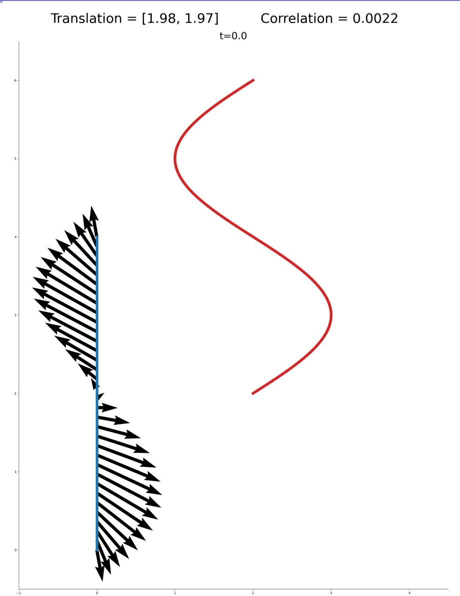

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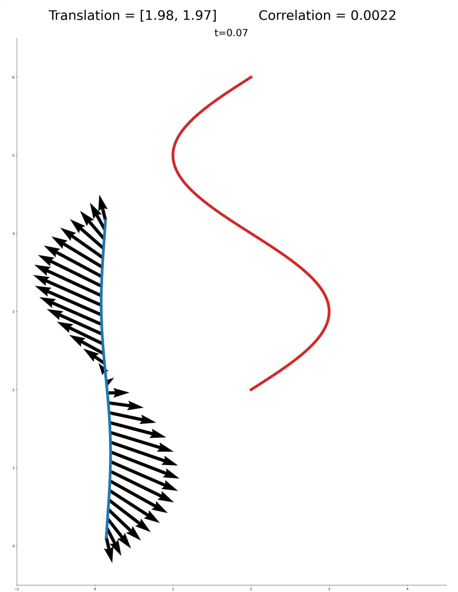

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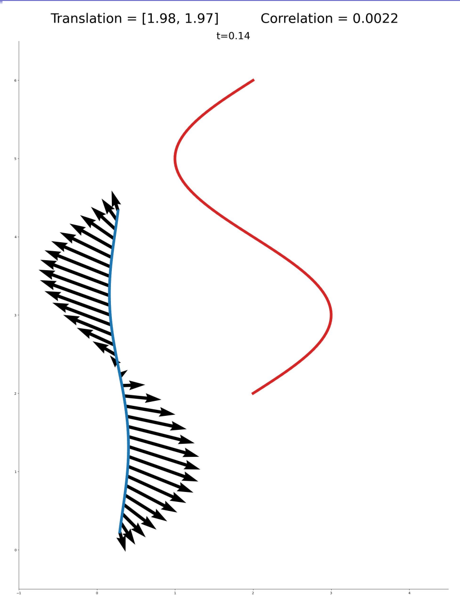

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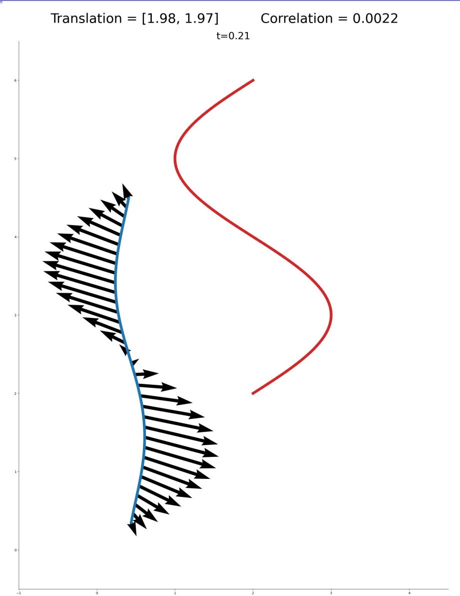

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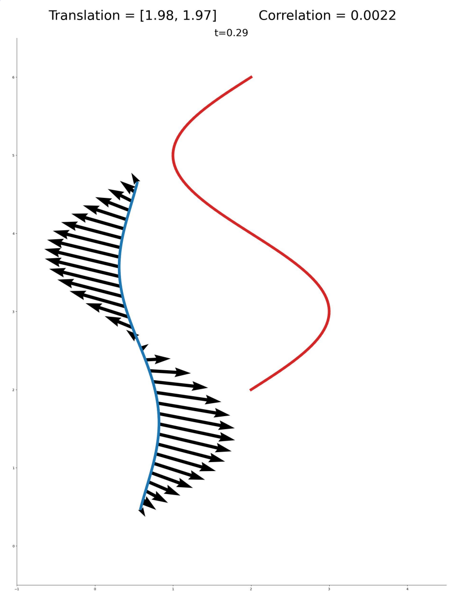

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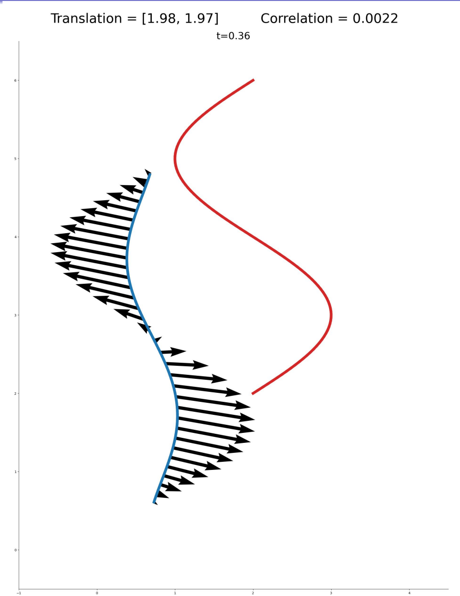

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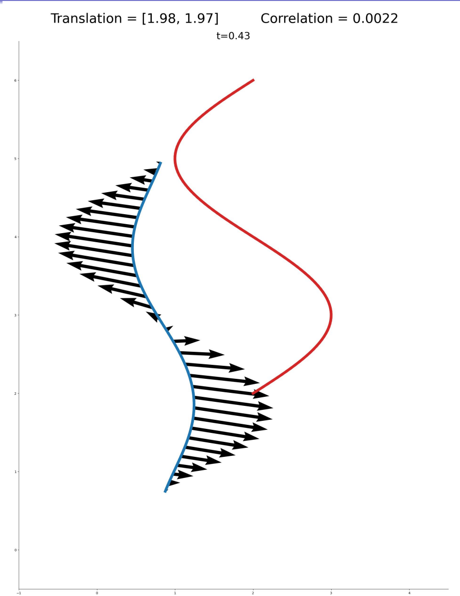

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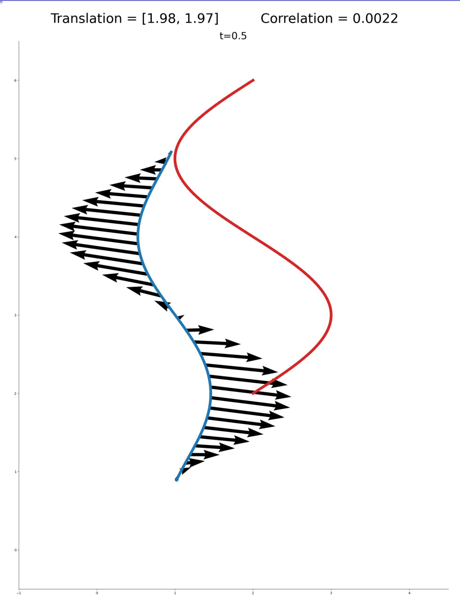

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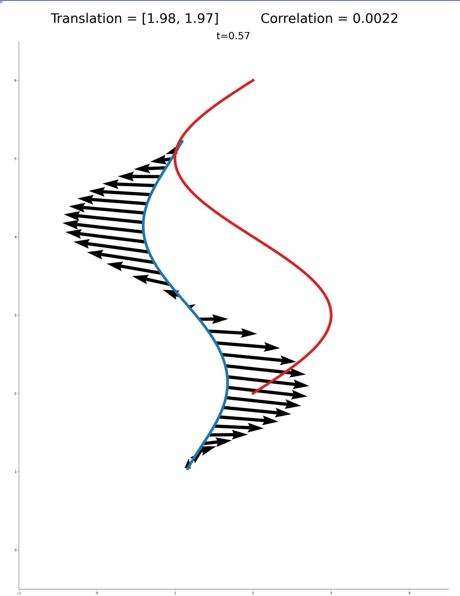

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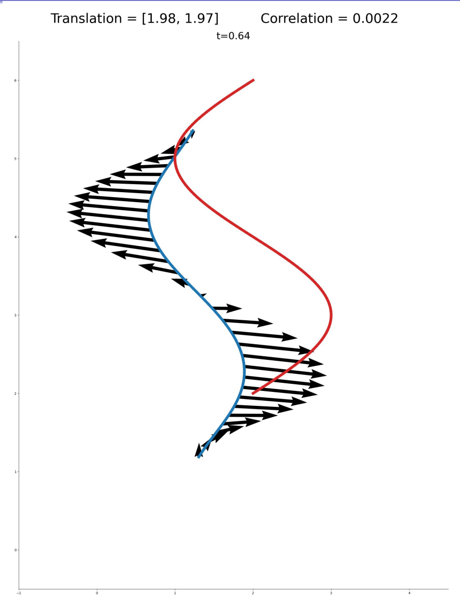

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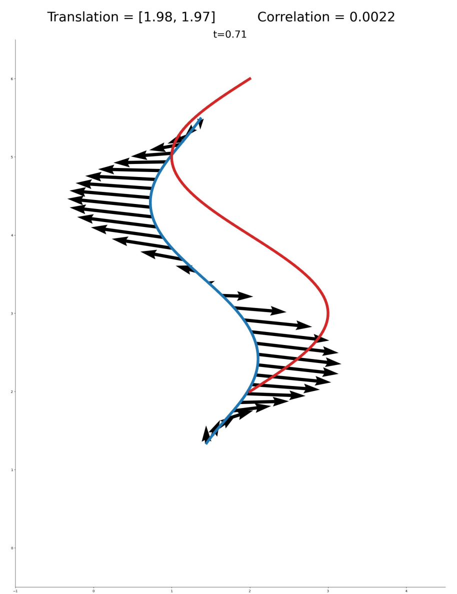

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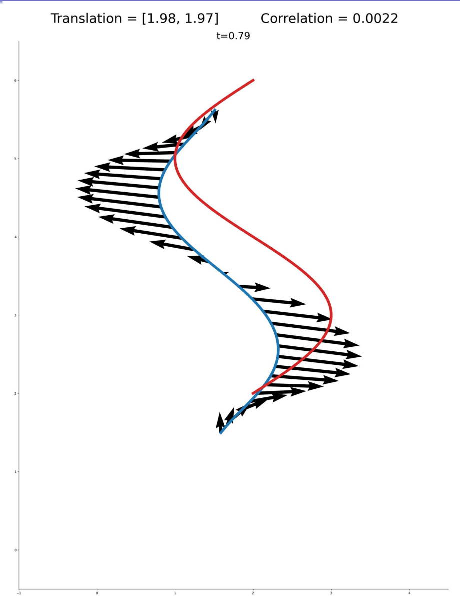

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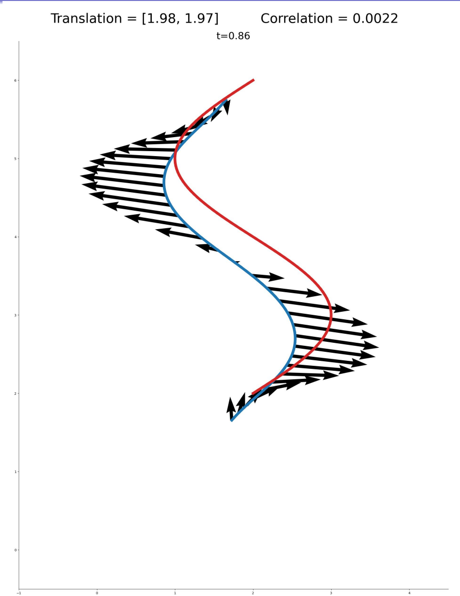

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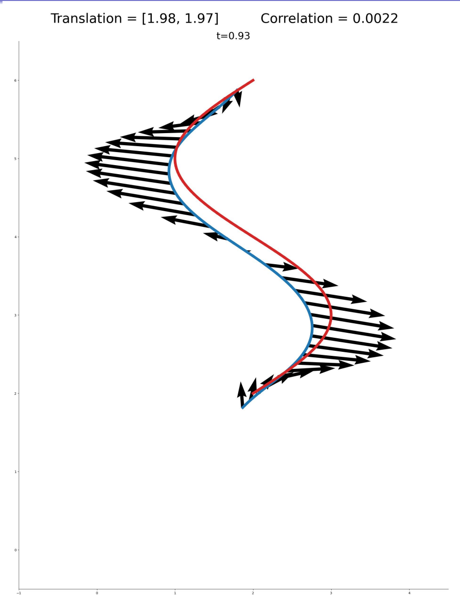

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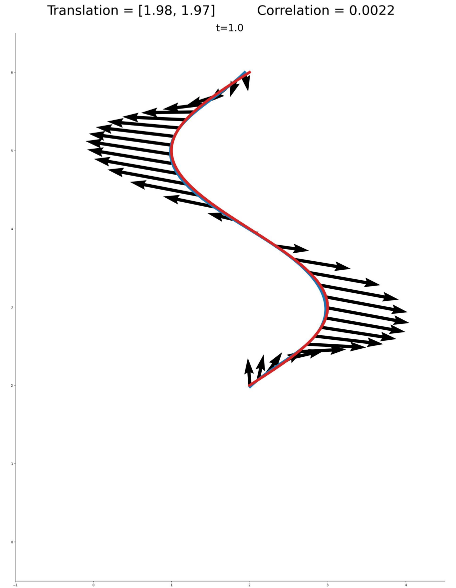

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