Introduction to shape analysis

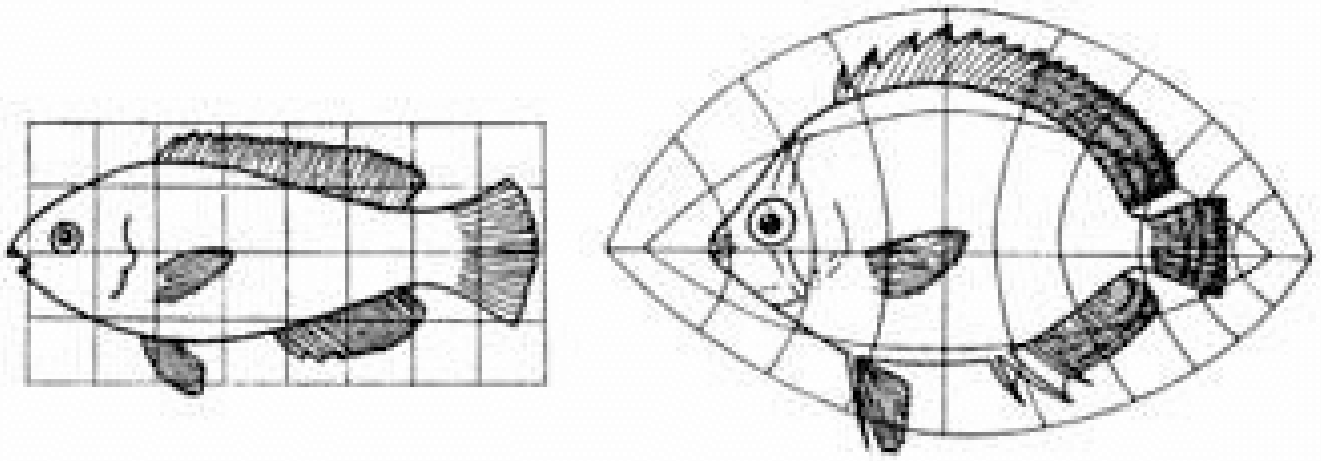
Special GTE

Rayane Mouhli

Introduction to shape analysis

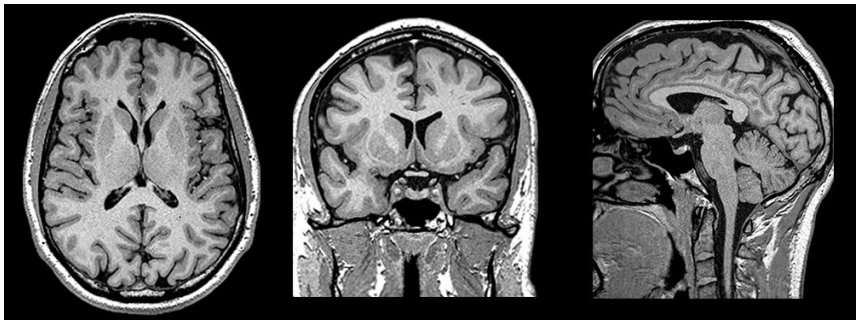
Main idea : Modelize and analyze the variability of biological shapes.

Computational anatomy introduced by the biologist D'Arcy Thompson (1917) in "On Growth and Form"

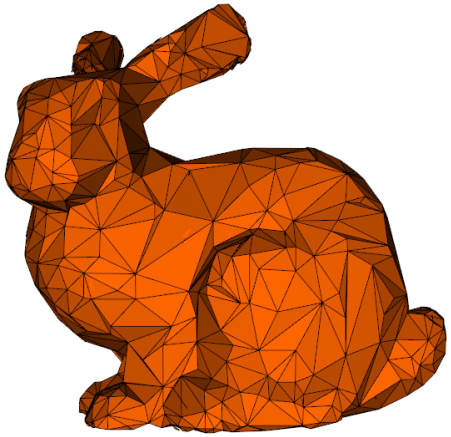


On Growth and Form - D'Arcy Thompson

Different types of data

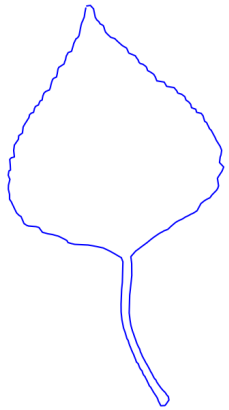


Images



Meshed surfaces

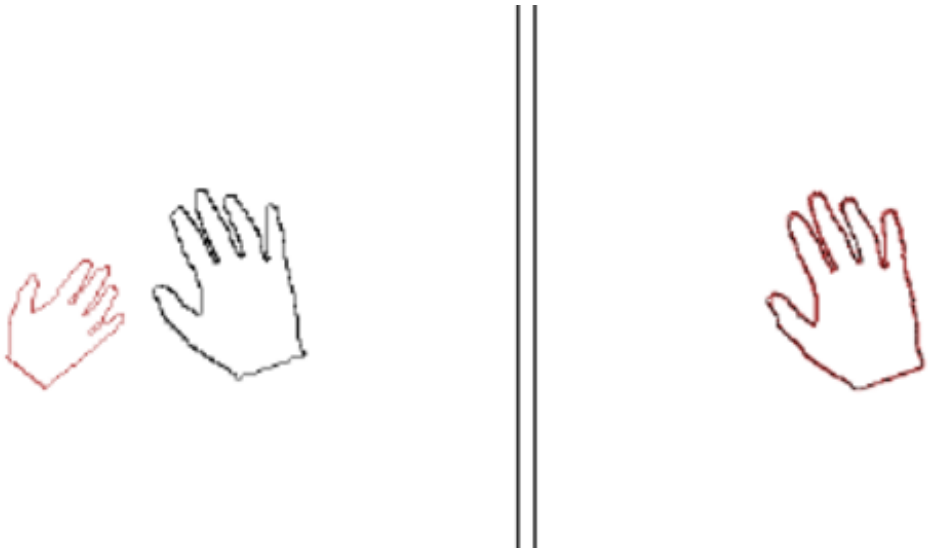
Landmarks



Curves

Registration/Matching

Given two objects $q^{(0)}, q^{(1)}$, we want to find the best deformation that transform the source $q^{(0)}$ into the target $q^{(1)}$.



Robust Rigid Shape Registration Method Using a Level Set Formulation - M.Al-Huseiny, S.Mahmoodi, M.Nixon

Shape spaces

Definition

A group of deformation is a Lie group G (i.e group + manifold).

Examples:

- Isometries (rotations + translation)s: $\mathrm{Isom}(\mathbb{R}^d) := \mathrm{SO}_d \ltimes \mathbb{R}^d$
- Diffeomorphisms: $\mathrm{Diff}(\mathbb{R}^d)$ (smooth, bijective maps whose inverse is also smooth)

Definition

A shape space is a manifold Q on which G can acts.

Examples:

- Landmarks: $(\mathbb{R}^d)^n$
- Curves: $\mathrm{Imm}([0,1], \mathbb{R}^d)$
- Images: $L^2(\Omega, \mathbb{R})$

Deformation of a template

Different examples of deformation through group actions:

$$\begin{array}{ccc} \mathrm{Diff}(\mathbb{R}^d) \times L^2(\Omega, \mathbb{R}) & \longrightarrow & L^2(\Omega, \mathbb{R}^d) \\ (\varphi, I) & \longmapsto & I \circ \varphi^{-1} \end{array}$$

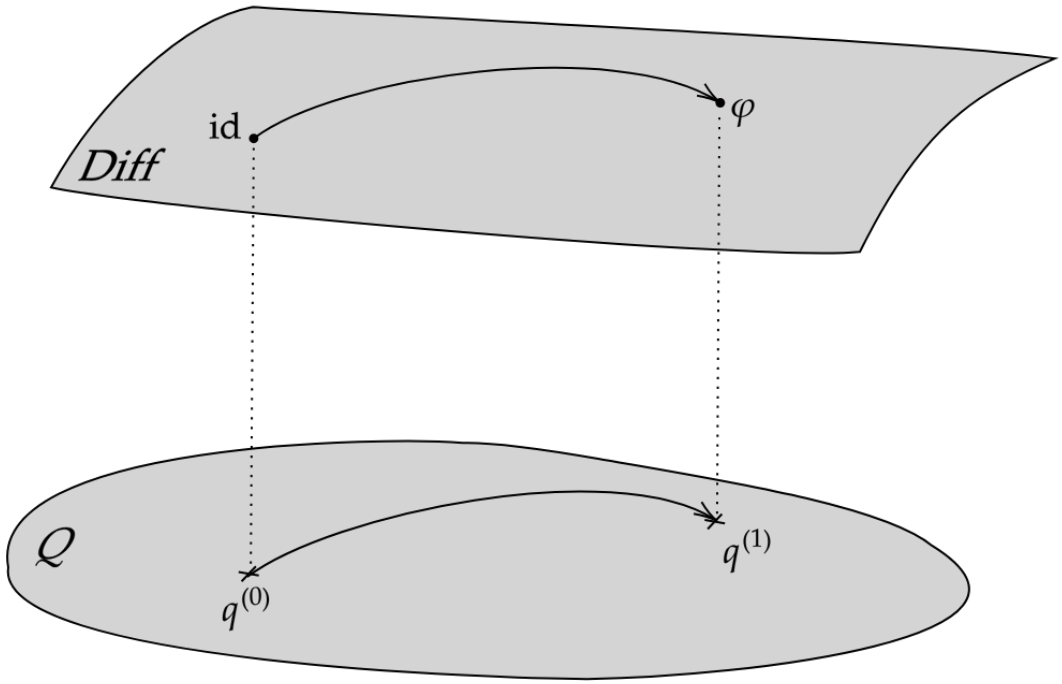
$$\begin{array}{ccc} \mathrm{Isom}(\mathbb{R}^d) \times L^2(\Omega, \mathbb{R}) & \longrightarrow & L^2(\Omega, \mathbb{R}^d) \\ ((R, T), I) & \longmapsto & I(R^\top(\cdot - T)) \end{array}$$

$$\begin{array}{ccc} (\mathrm{Isom}(\mathbb{R}^d) \times \mathrm{Diff}(\mathbb{R}^d)) \times (\mathbb{R}^d)^n & \longrightarrow & (\mathbb{R}^d)^n \\ ((R, T, \varphi), (x_i)_i) & \longmapsto & (R\varphi(x_i) + T)_i \end{array}$$

Distance in the shape space

Shortest path between two shapes $q^{(0)}$ and $q^{(1)}$ $\Longrightarrow$ Notion of distance on $\mathcal{Q}$.

$$d_{\mathcal{Q}}(q^{(0)}, q^{(1)}) = \inf_{\varphi \in \mathrm{Diff}(\mathbb{R}^d)} \{d(\mathrm{id}, \varphi) \mid \varphi \cdot q^{(0)} = q^{(1)}\}$$



How can we define a distance on $\mathrm{Diff}(\mathbb{R}^d)$?

Large Deformation Diffeomorphic Metric Mapping[1]

Theorem

Let $v \in L^2([0,1], V)$ be a time-varying vector field with $V \hookrightarrow \mathcal{C}_0^2(\mathbb{R}^d, \mathbb{R}^d)$. The flow of diffeomorphism ϕ^v generated by v is the unique solution of :

$$\phi_t^v = \mathrm{id} + \int_0^t v_s \circ \phi_s^v ds$$

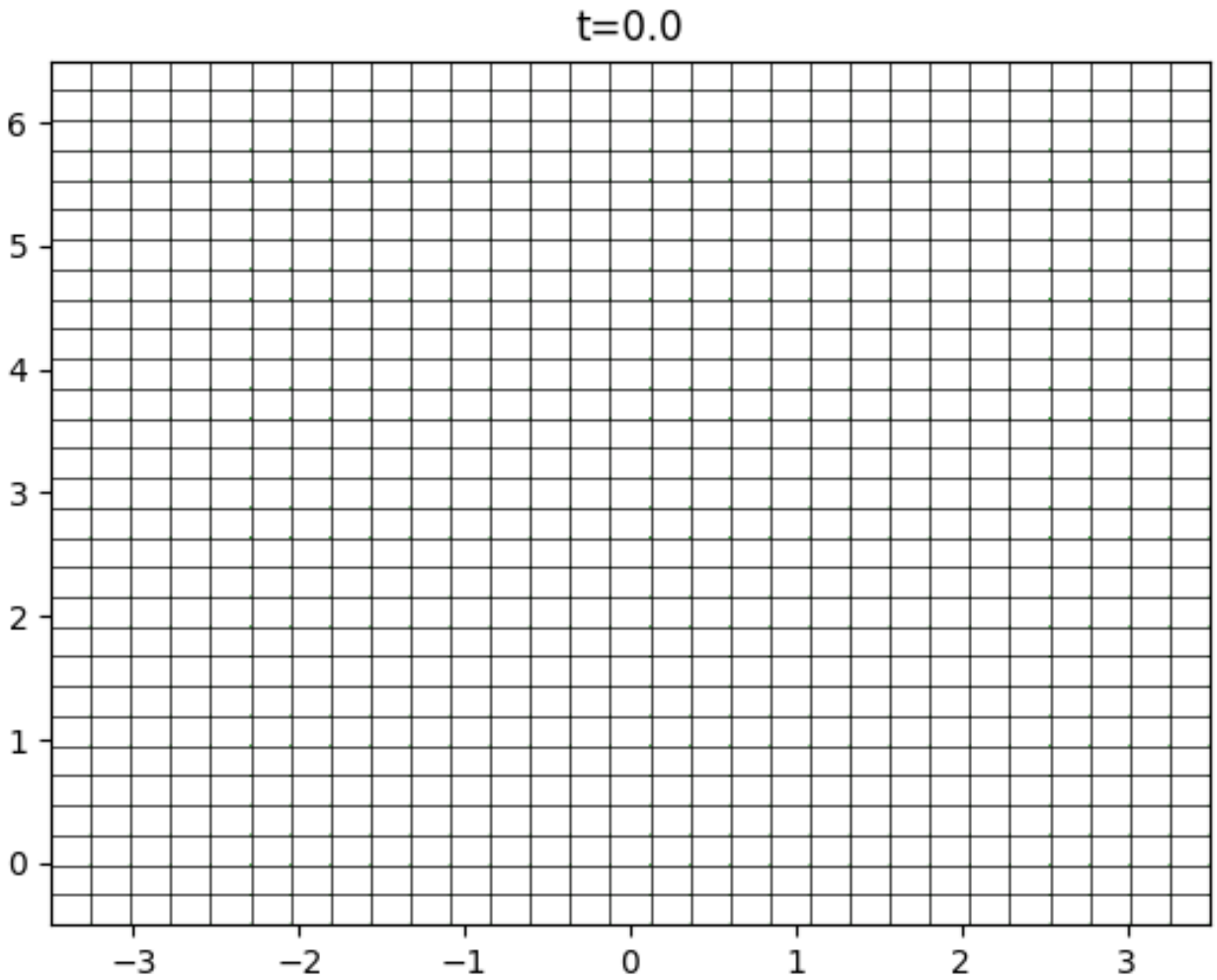
The distance on the space of diffeomorphisms is defined by :

$$d(\mathrm{id}, \varphi) = \inf_{v \in L^2([0,1],V)} \{\|v\|_{L^2} \mid \varphi = \phi_1^v\}$$

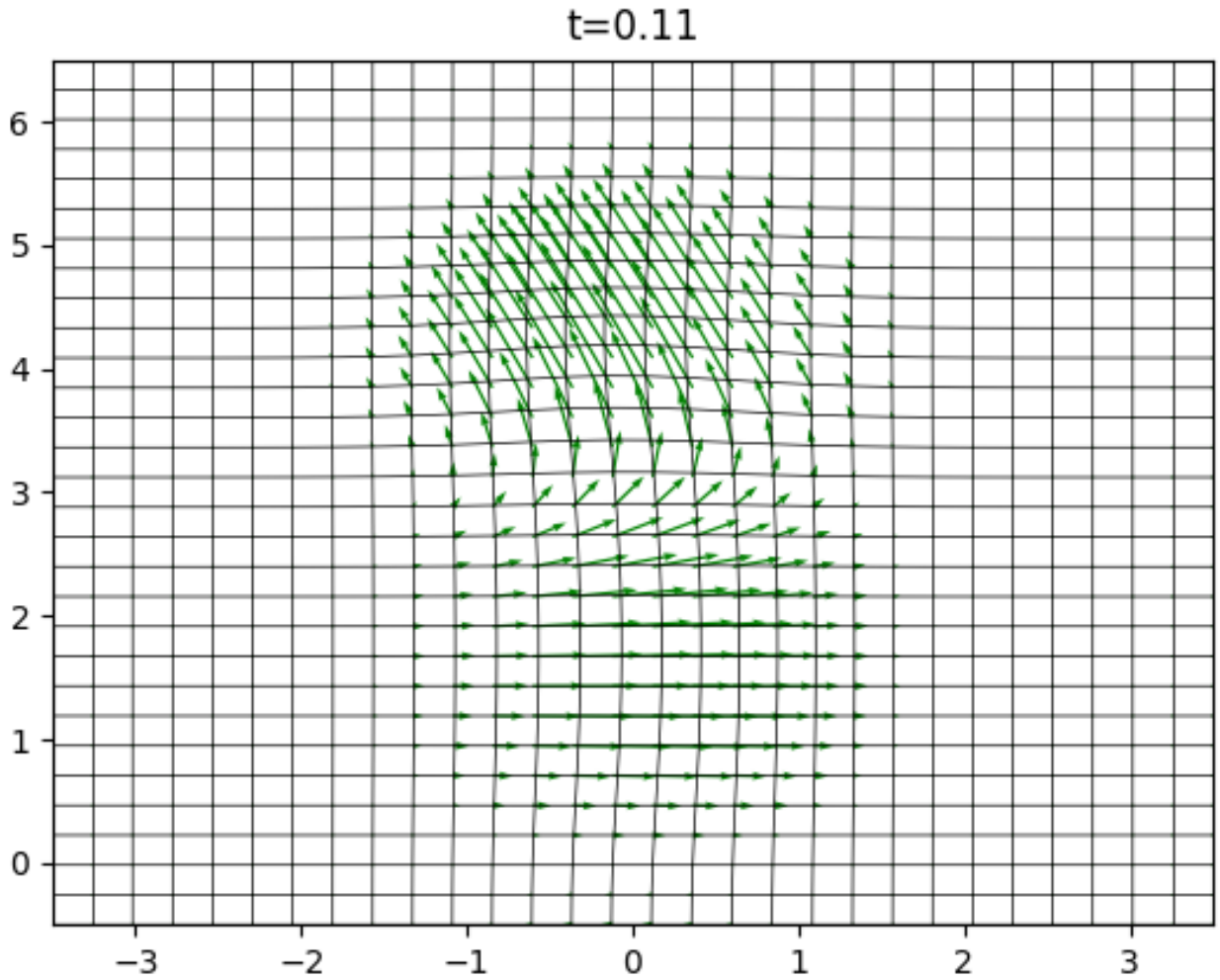
In practice, V is a Reproducing Kernel Hilbert Space (RKHS).

[1] Beg, Miller, Trouvé, Younes 2005

Diffeomorphism generated by a vector field



Diffeomorphism generated by a vector field



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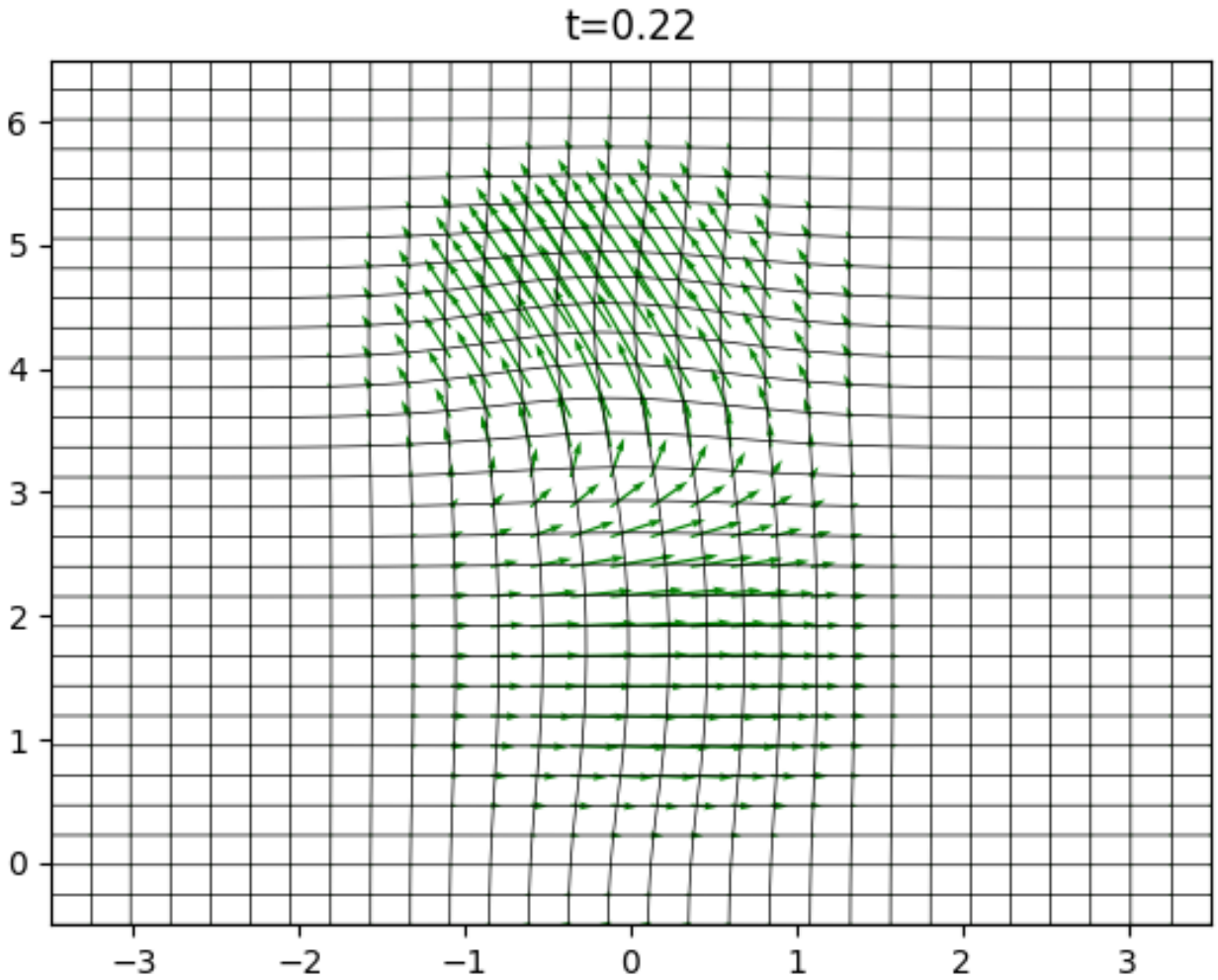

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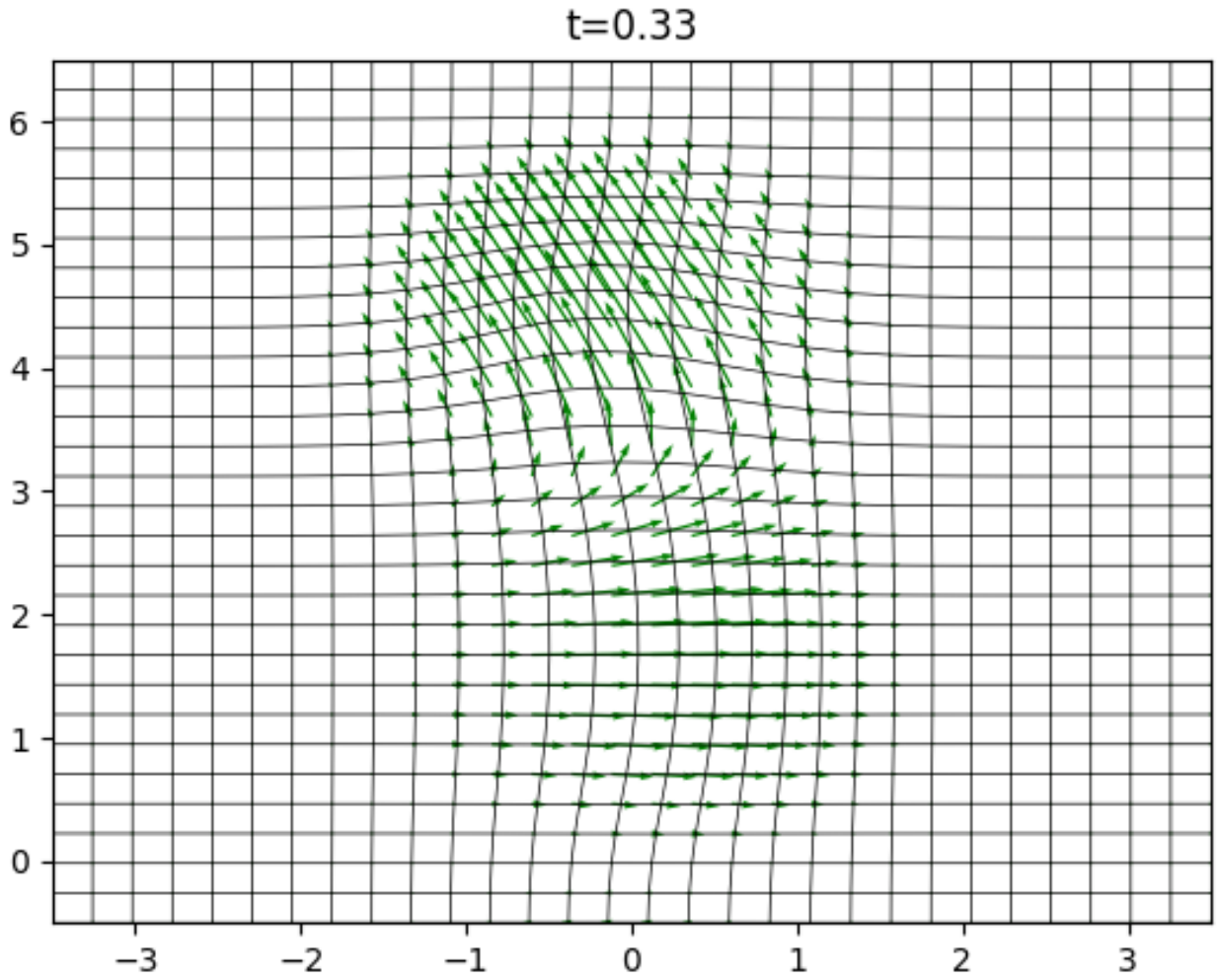

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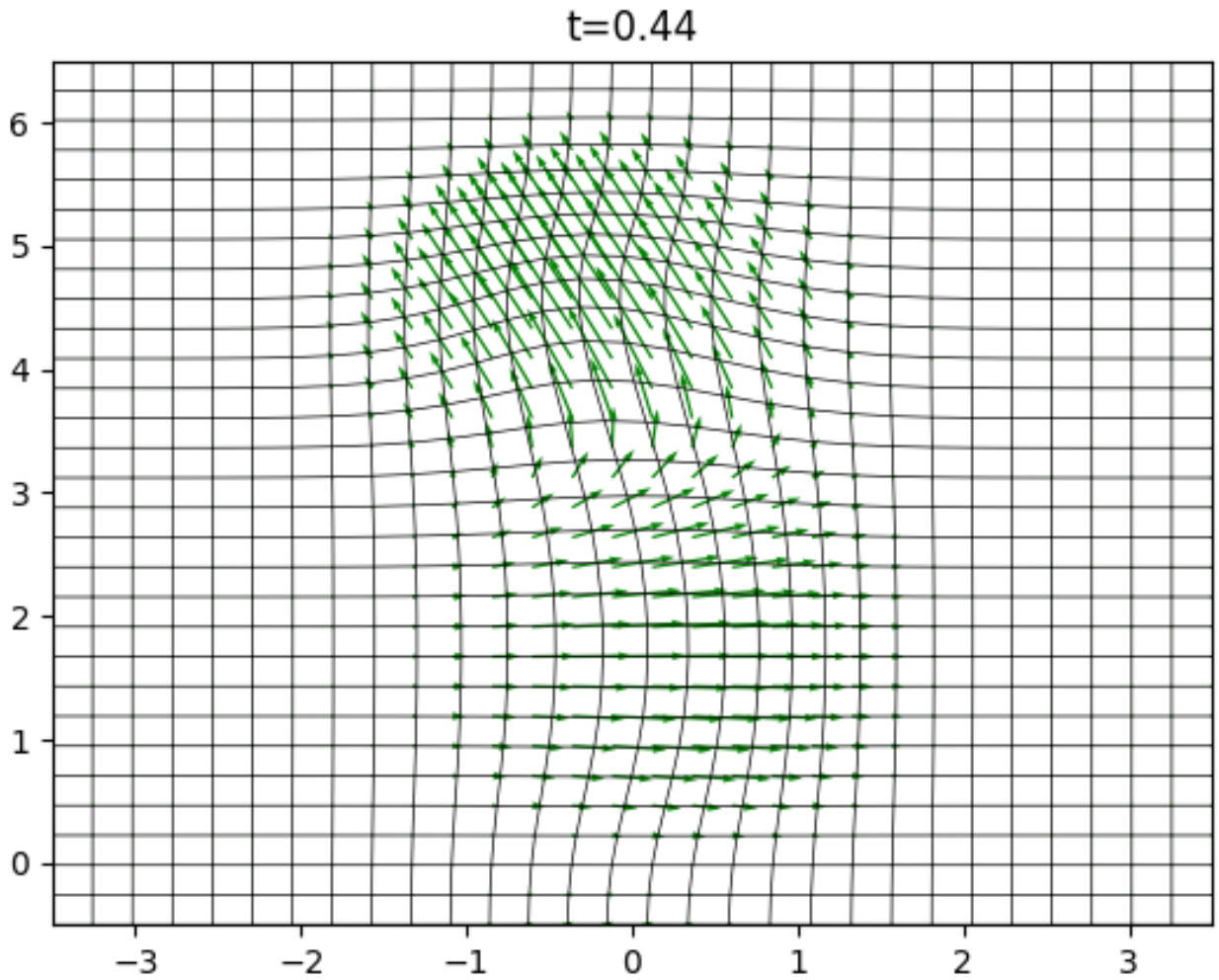

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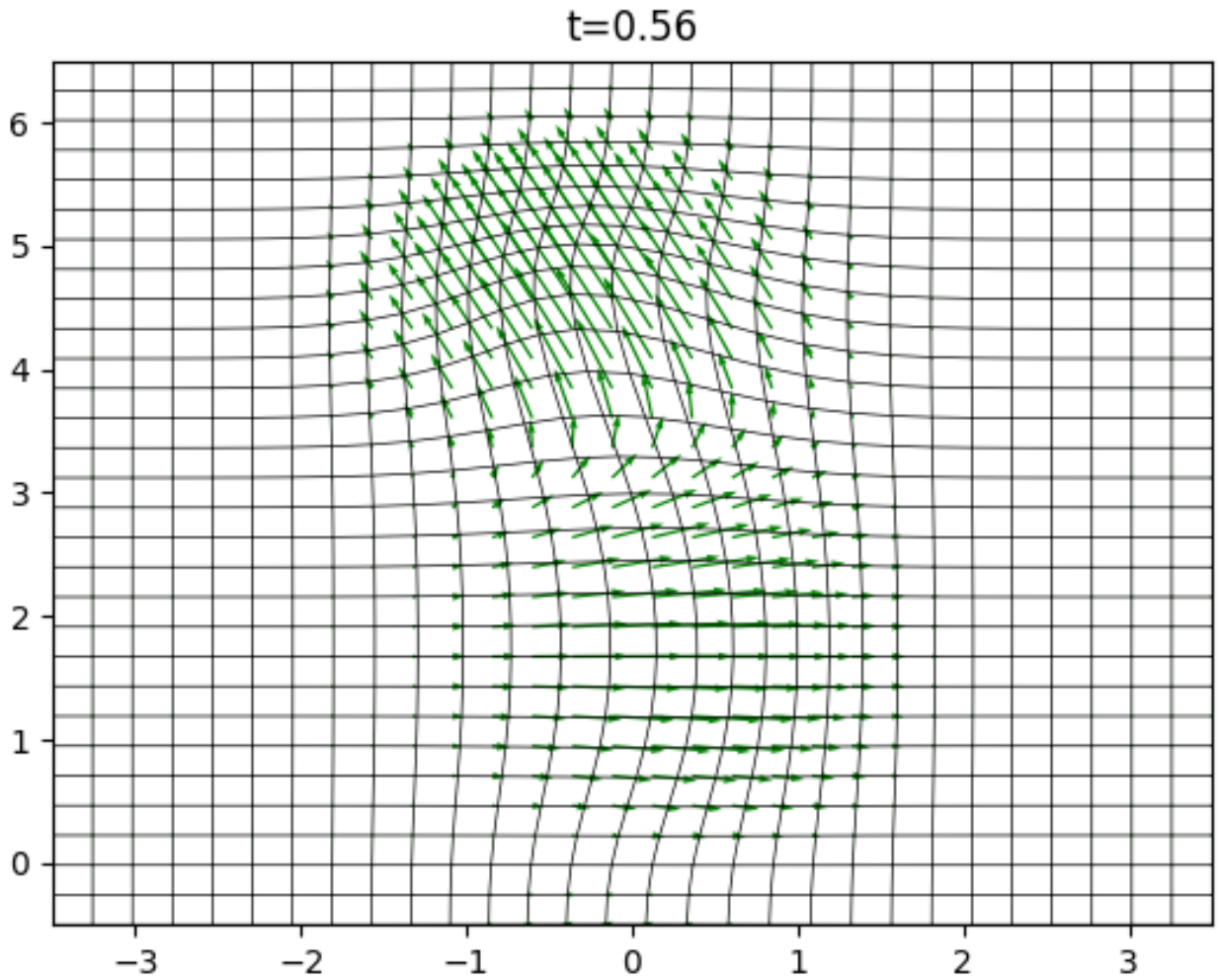

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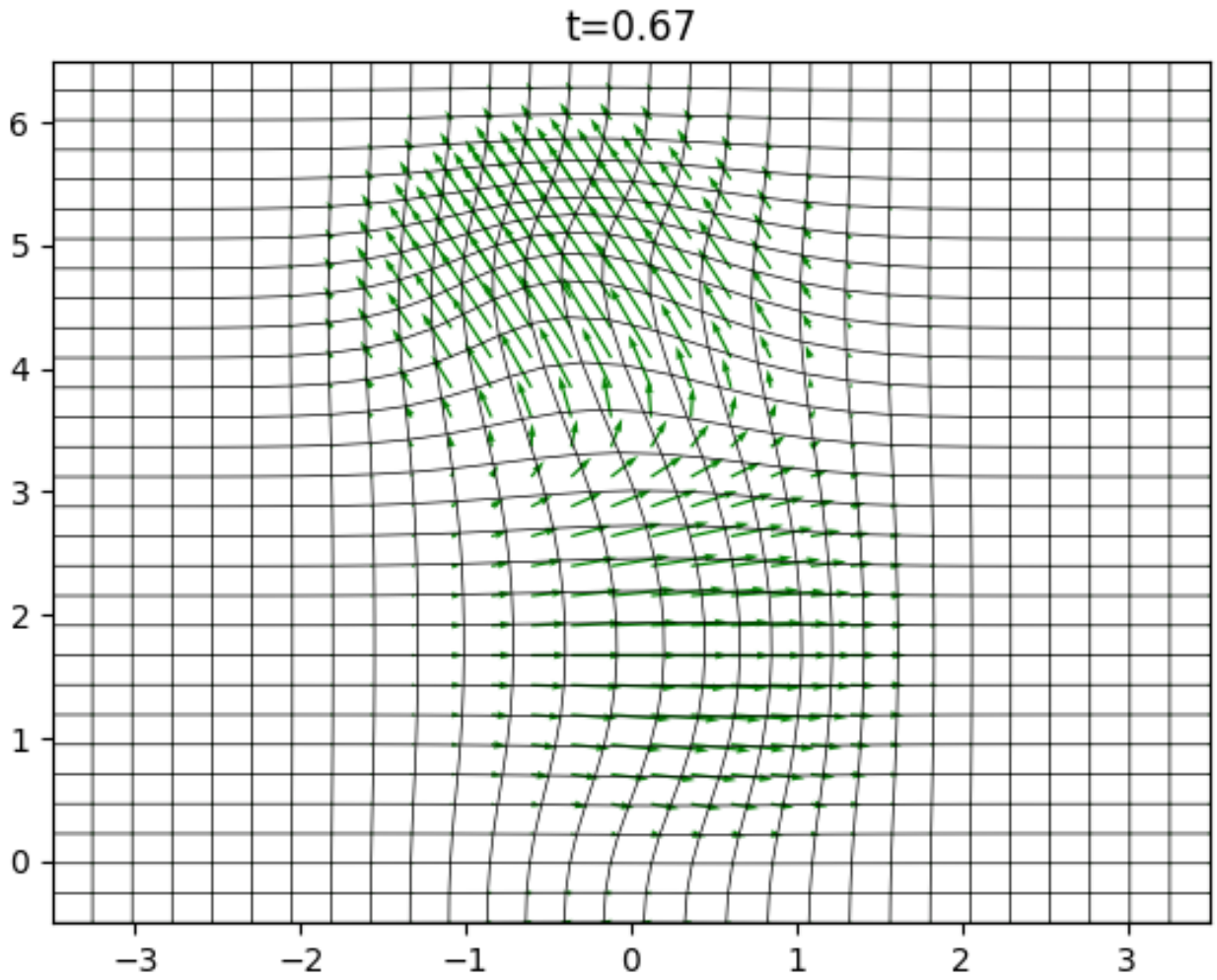

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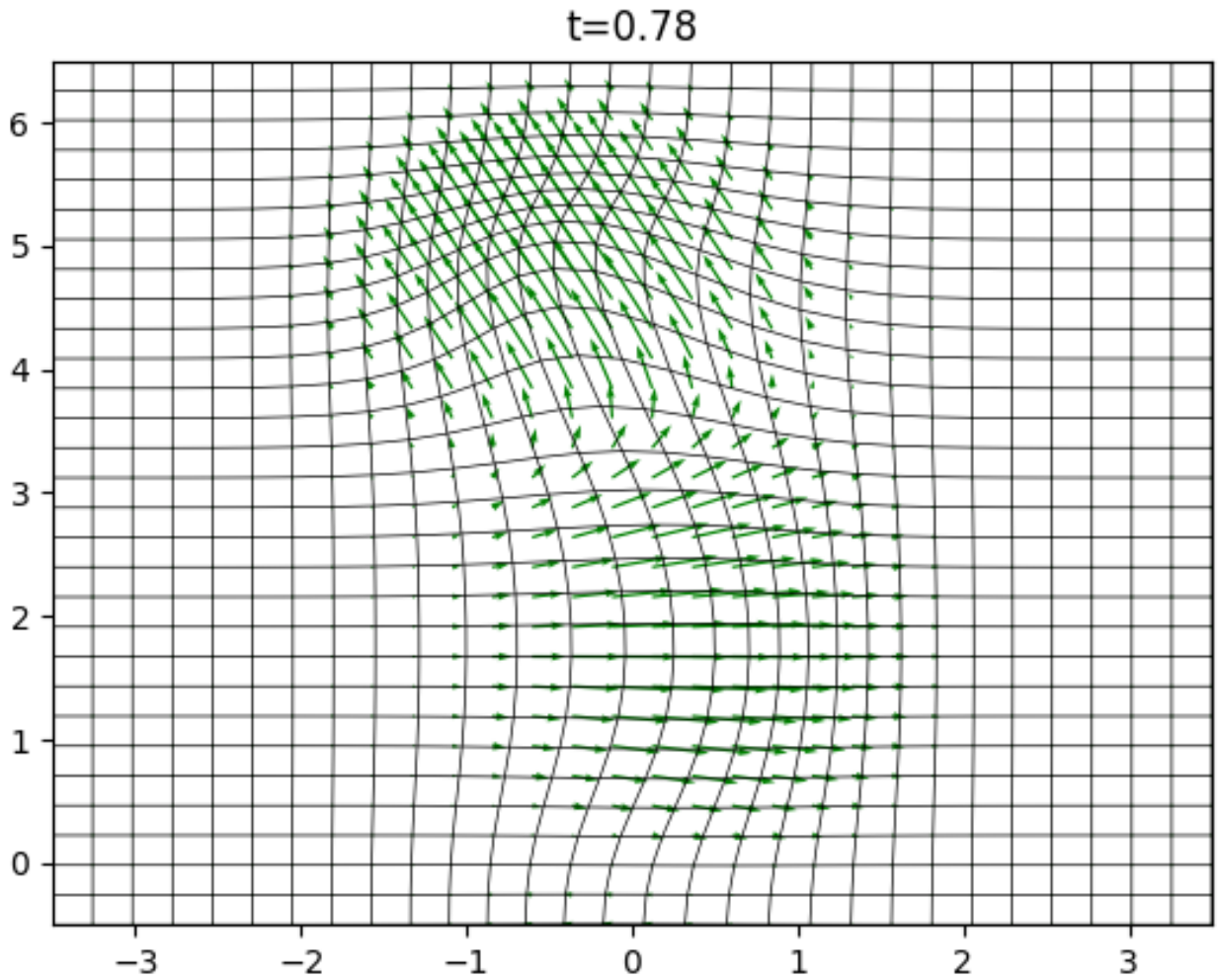

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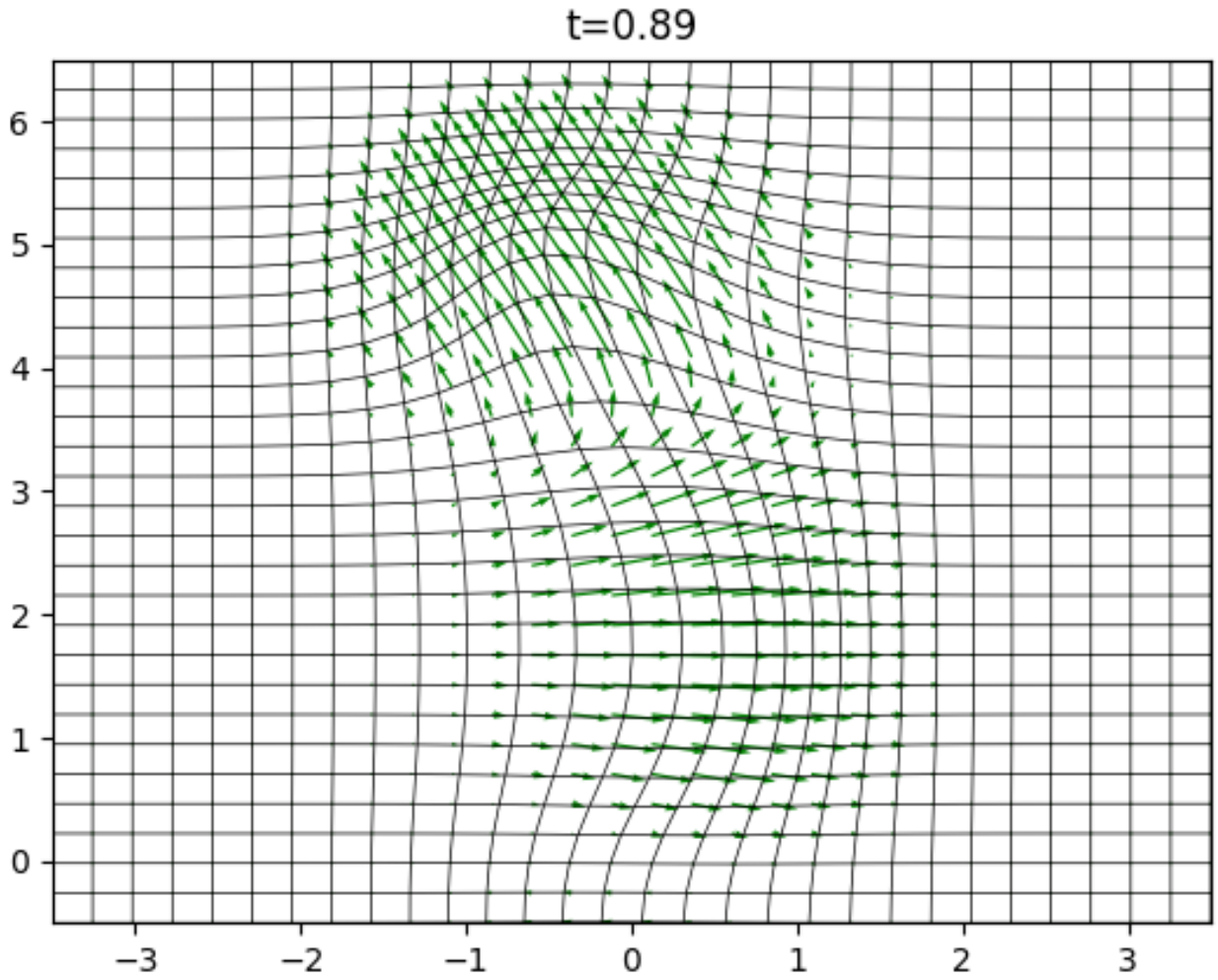

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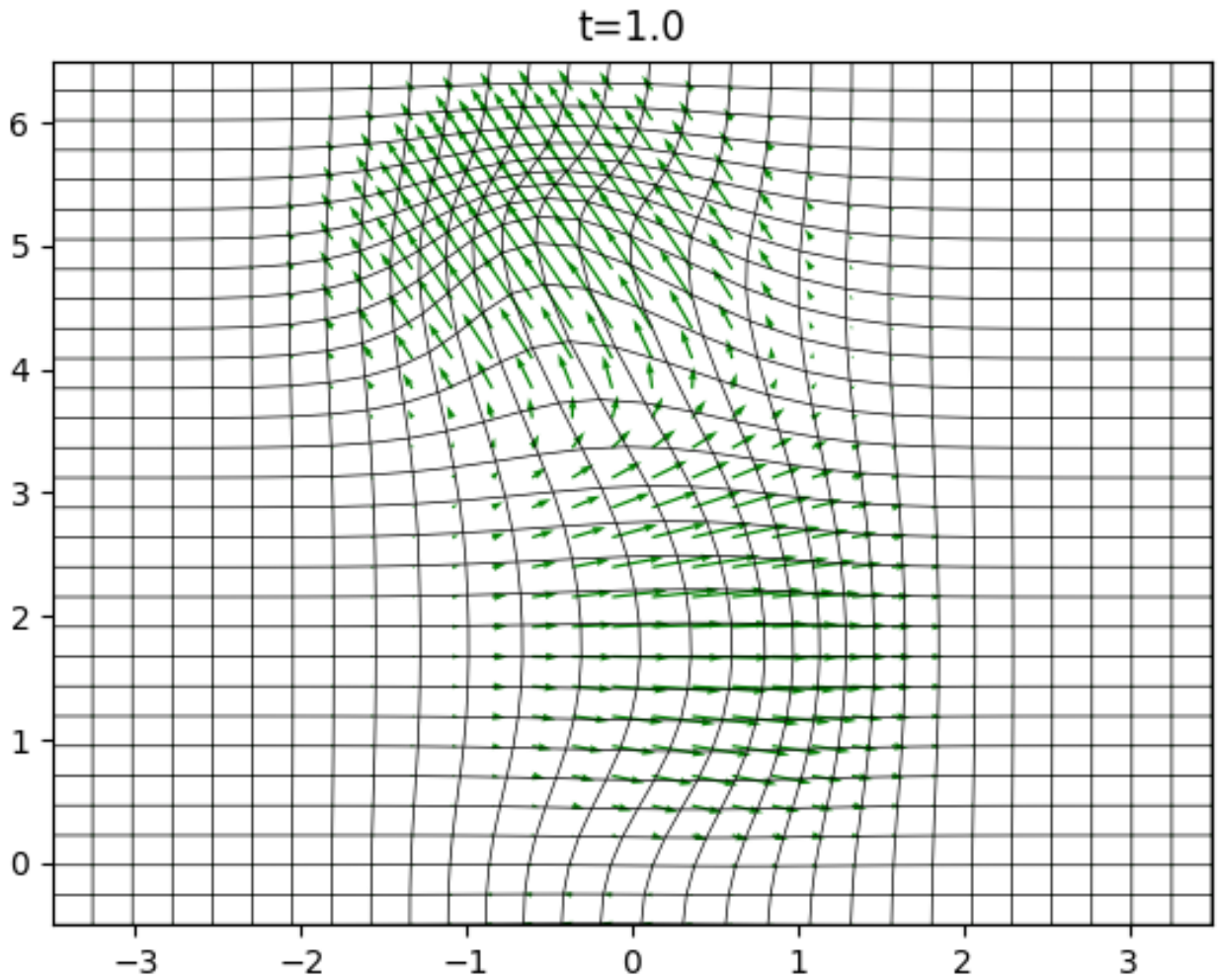
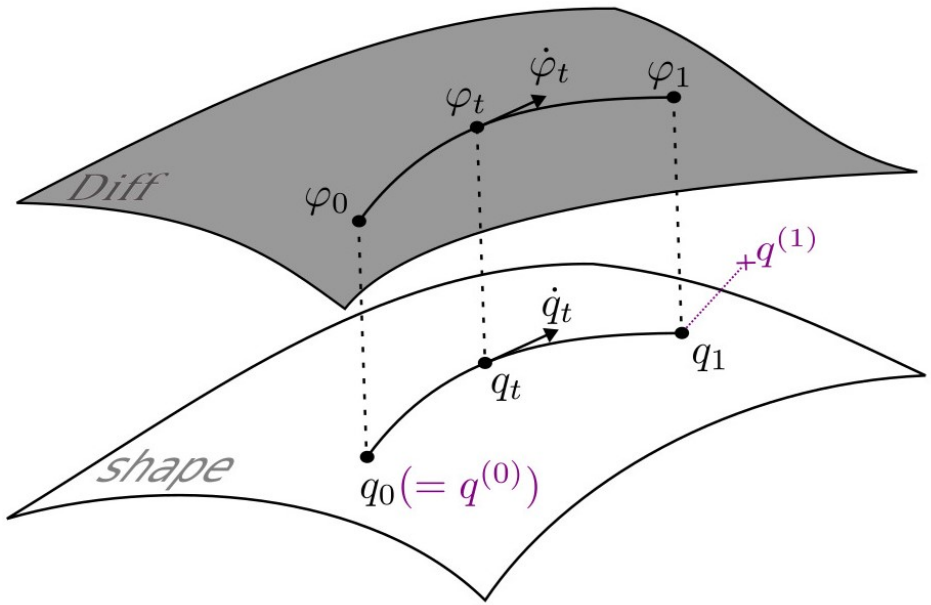


Inexact matching

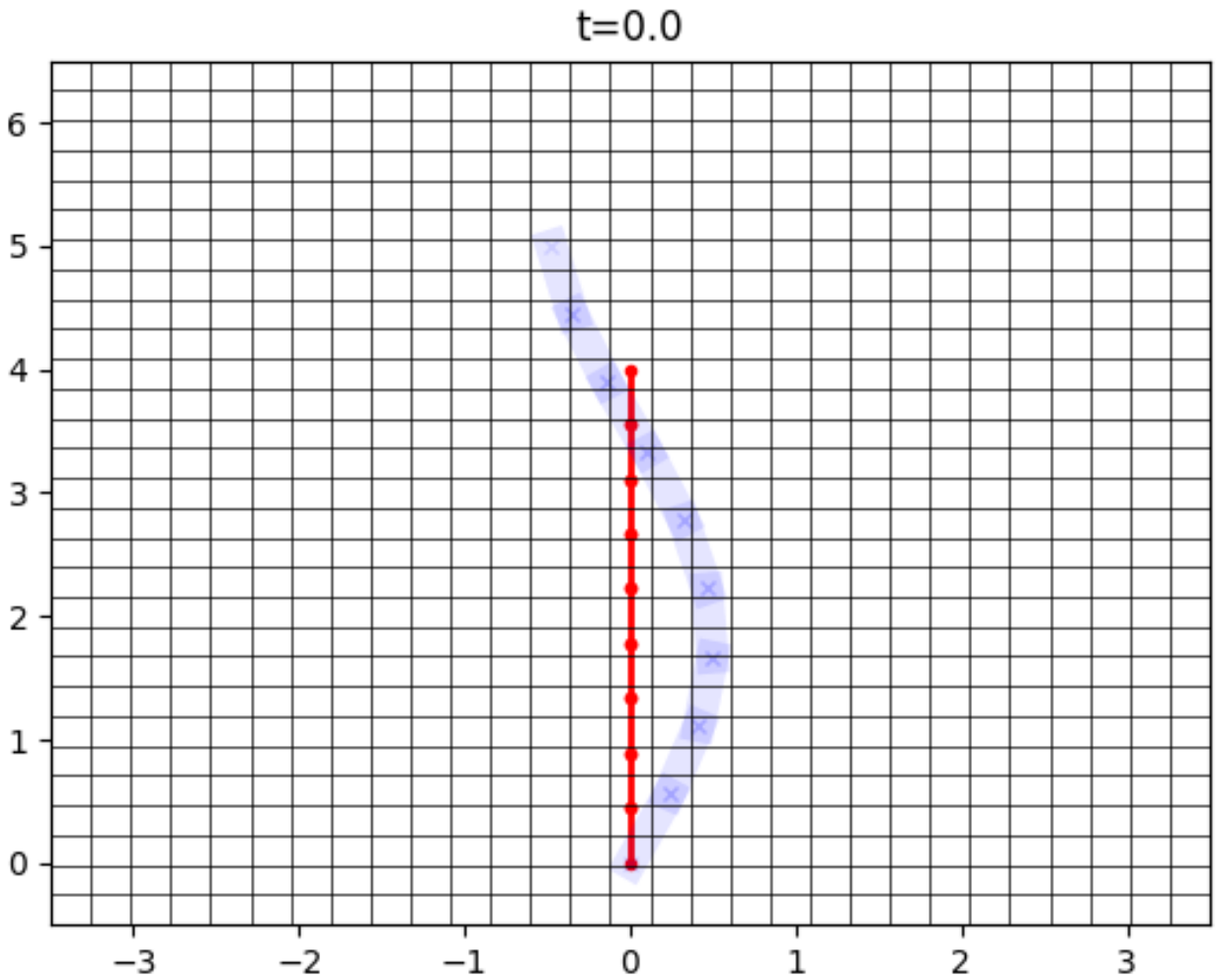
Shape registration corresponds to the following energy minimization problem:

$$\min_{v \in L^2([0,1],V)} E(v) = \int_0^1 \frac{1}{2}|v_t|_V^2 \, dt + D(\varphi_1 \cdot q^{(0)}, q^{(1)})$$

$$\text{subject to } \begin{cases} \dot{q}_t &= v_t \cdot q_t \\ q_0 &= q^{(0)} \end{cases}$$



Shape deformation



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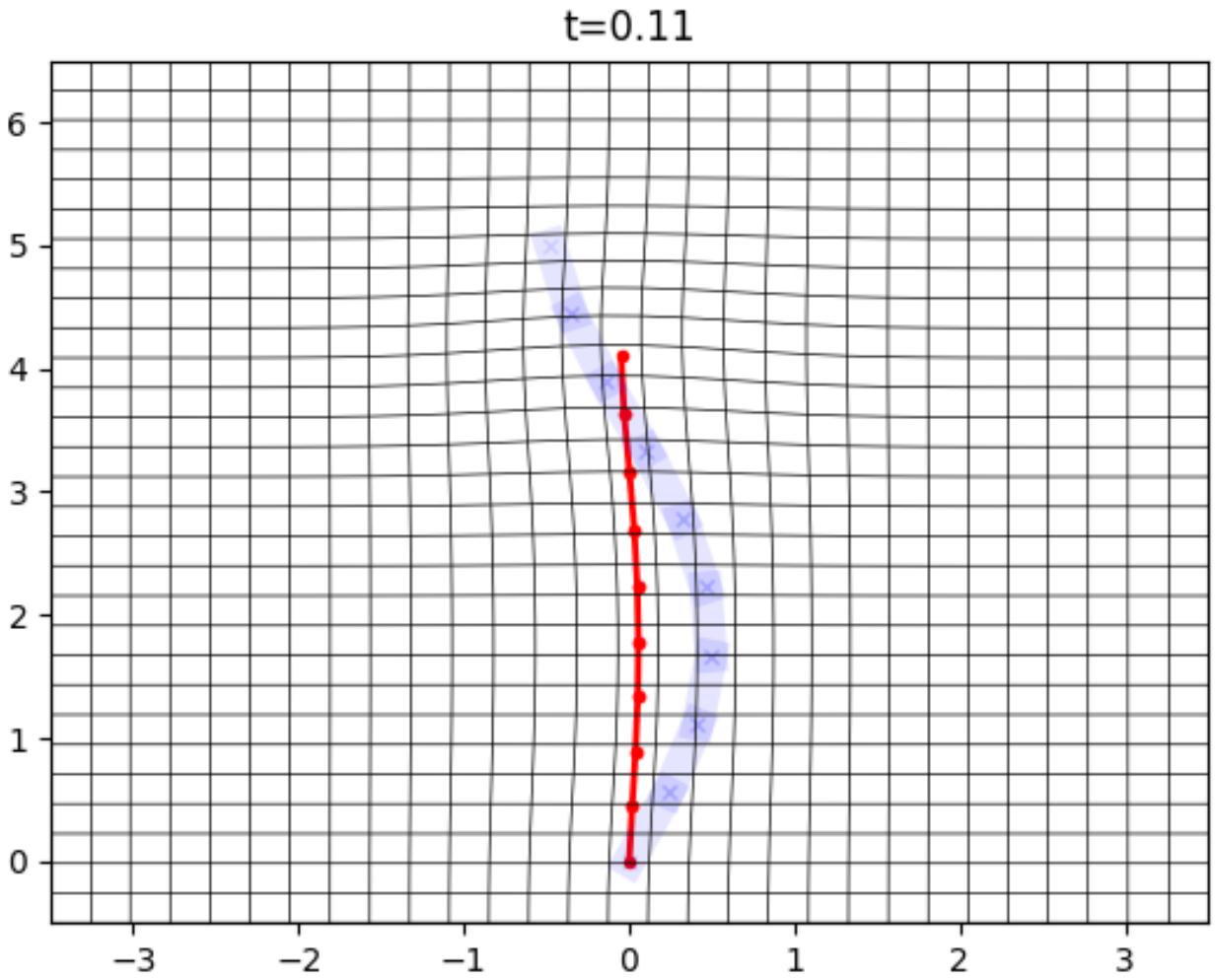
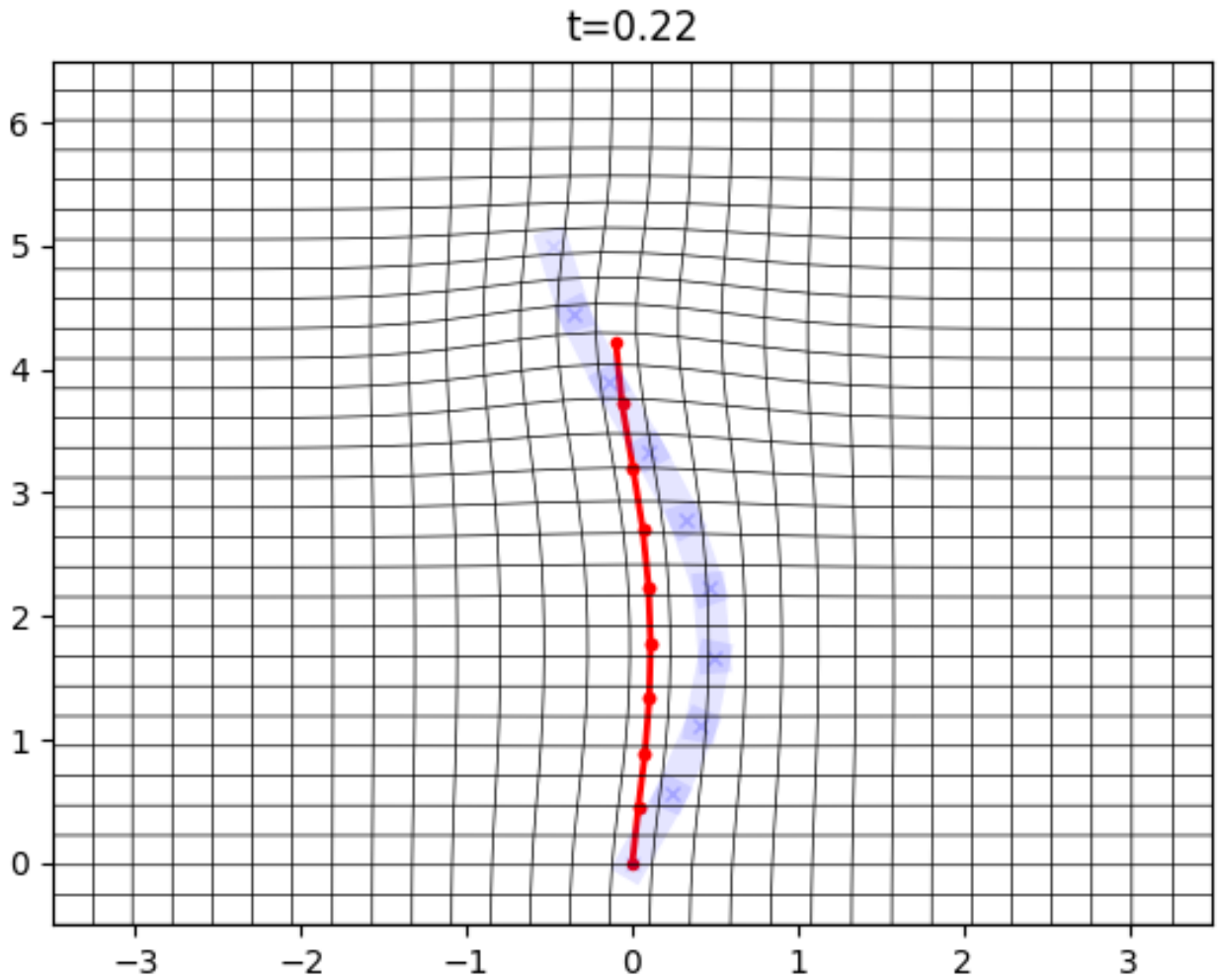


Shape deformation



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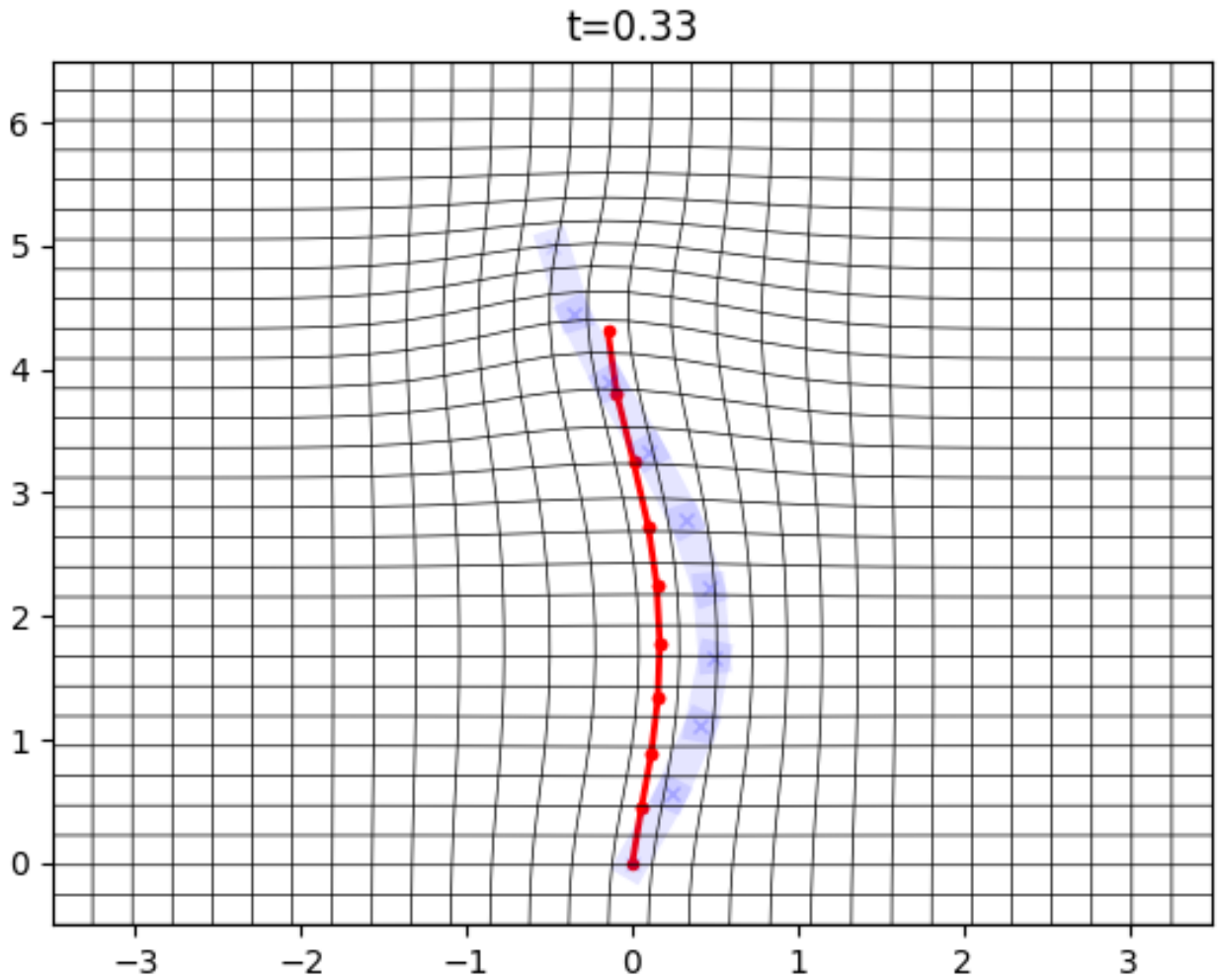
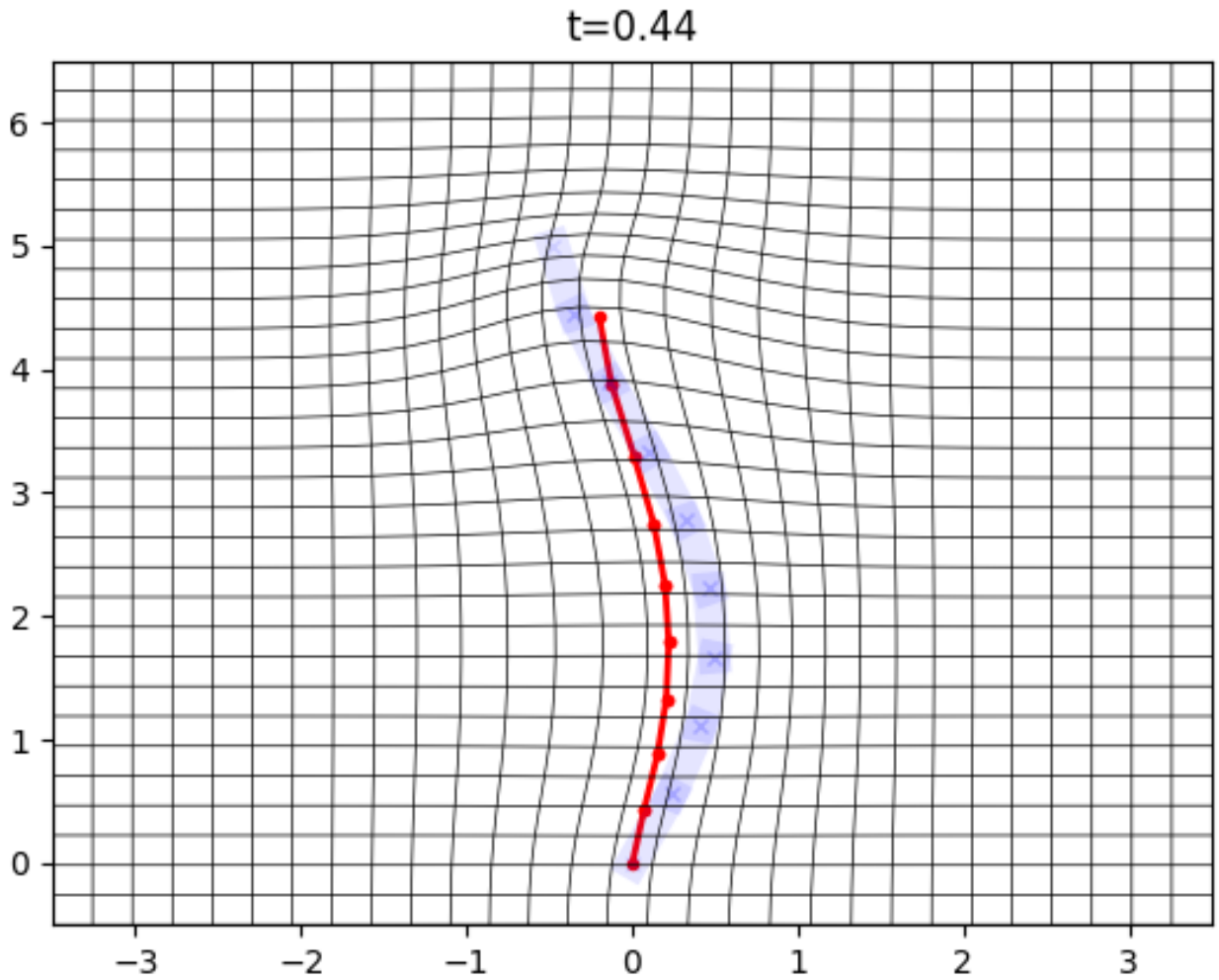


Shape deformation



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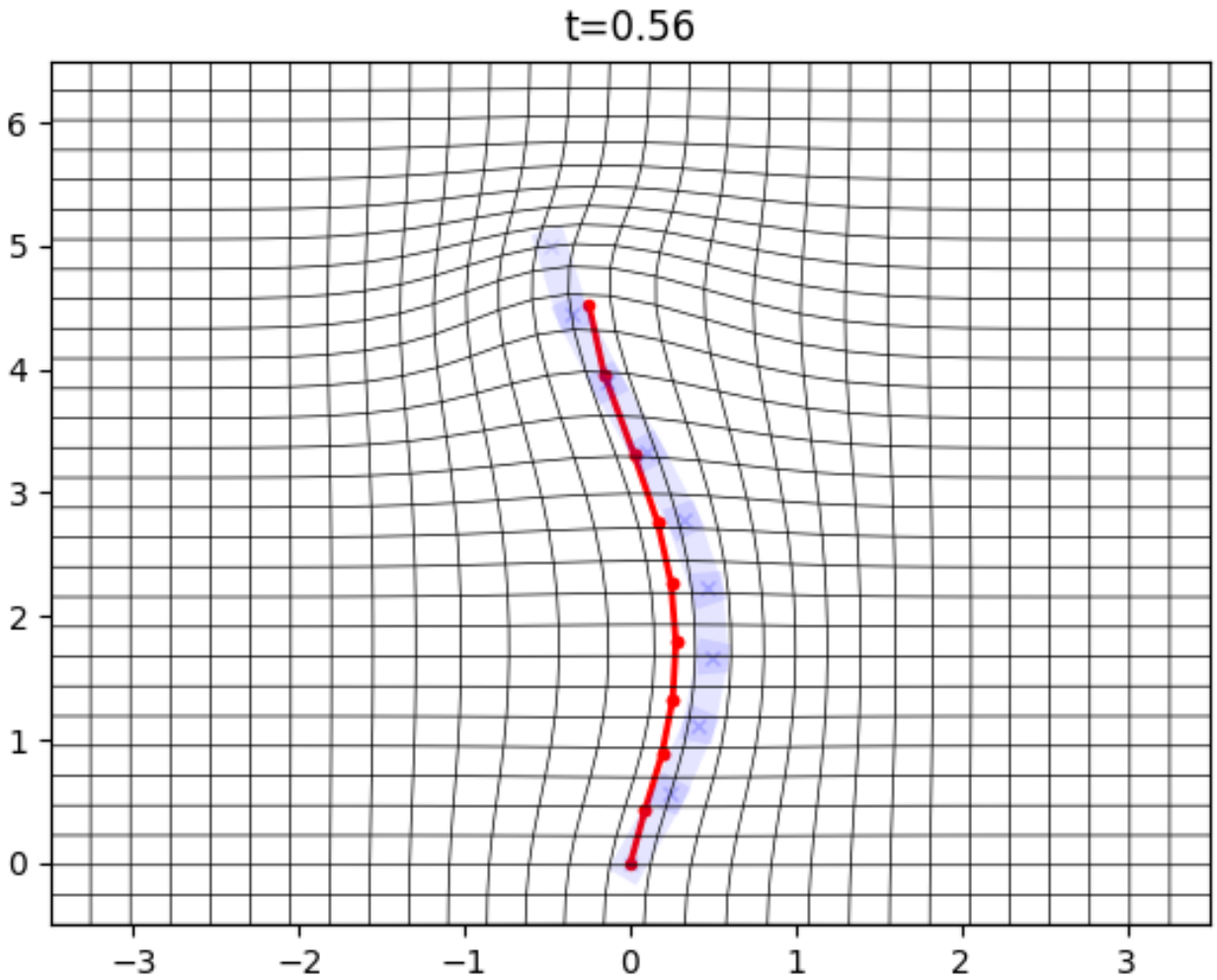
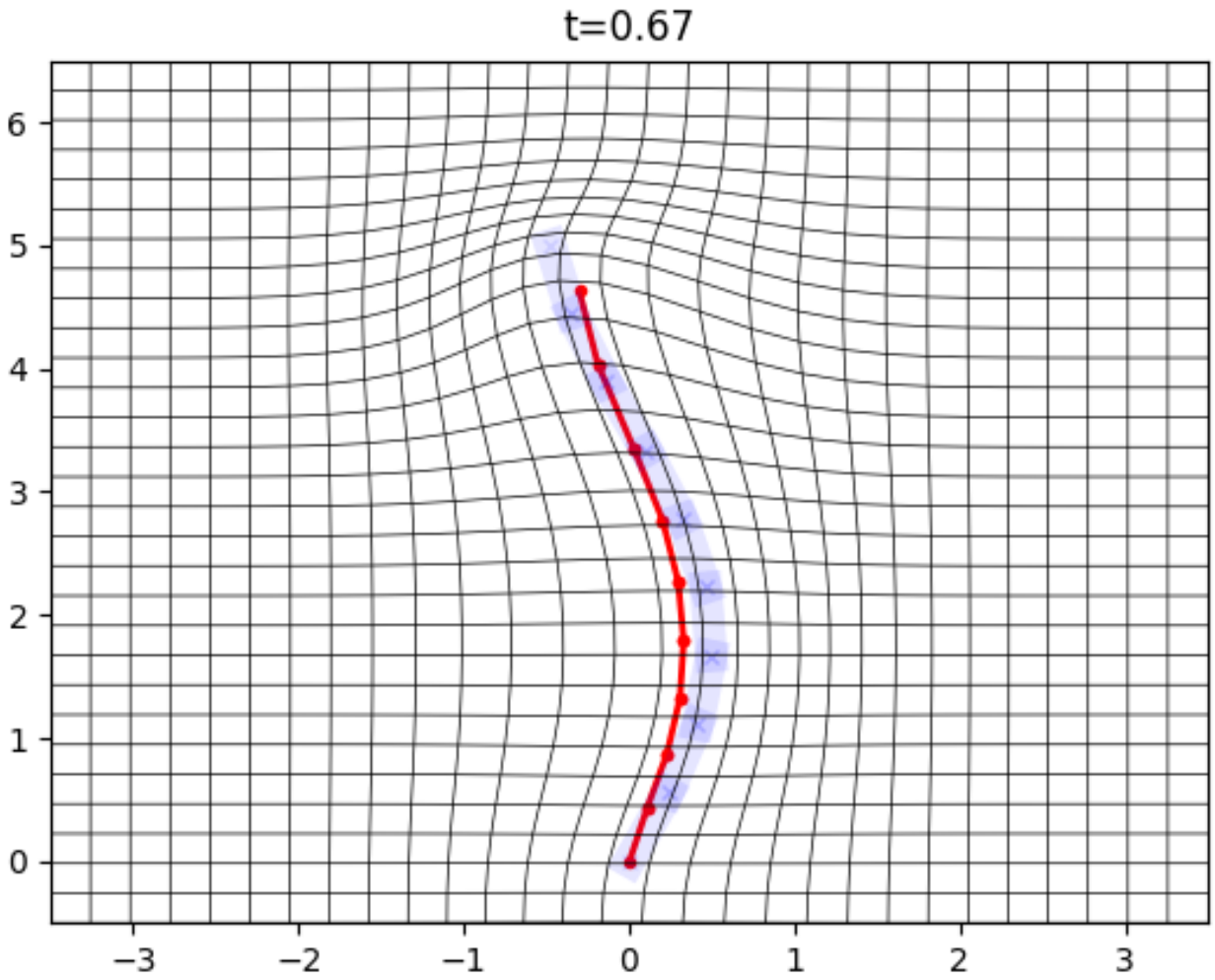


Shape deformation



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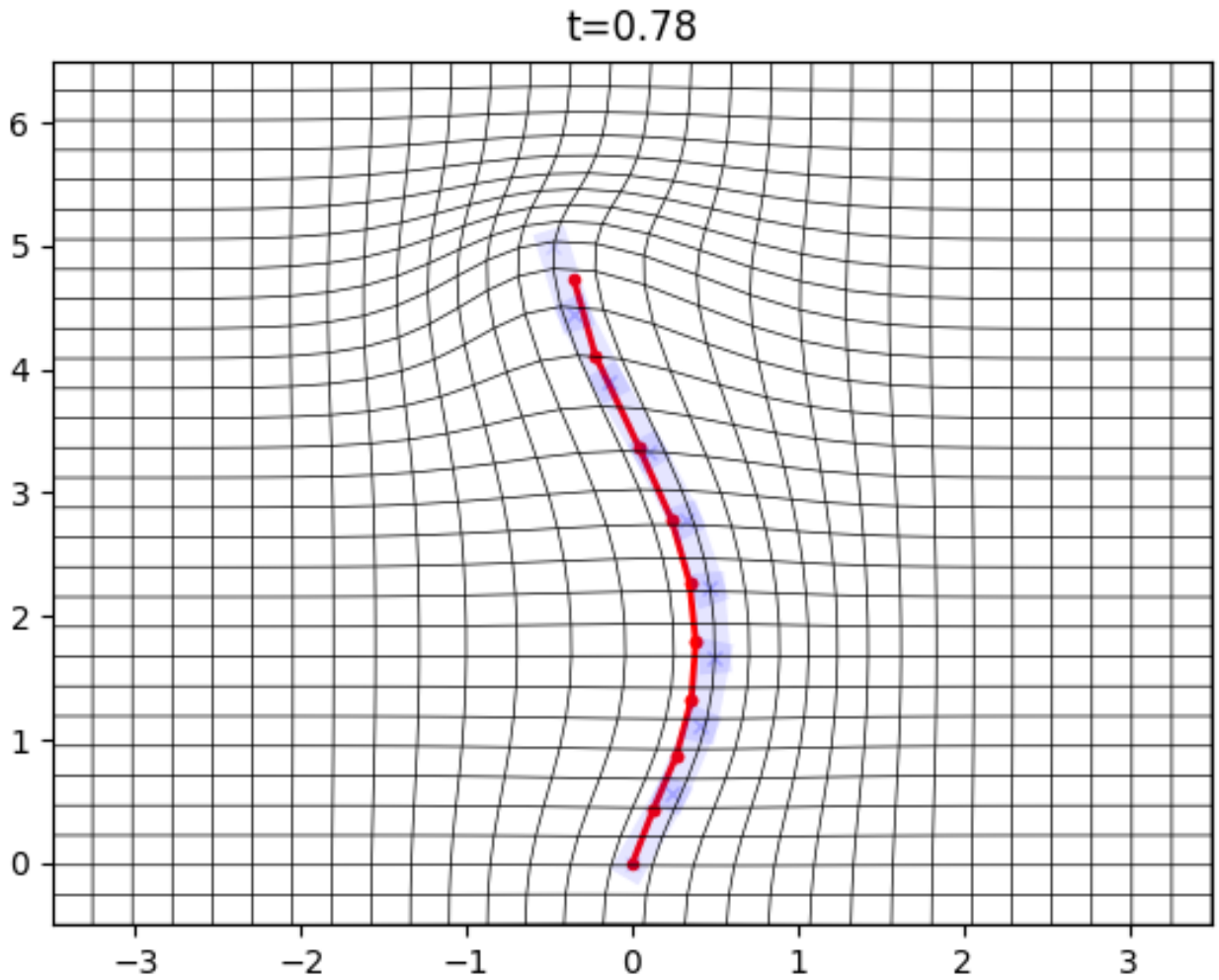
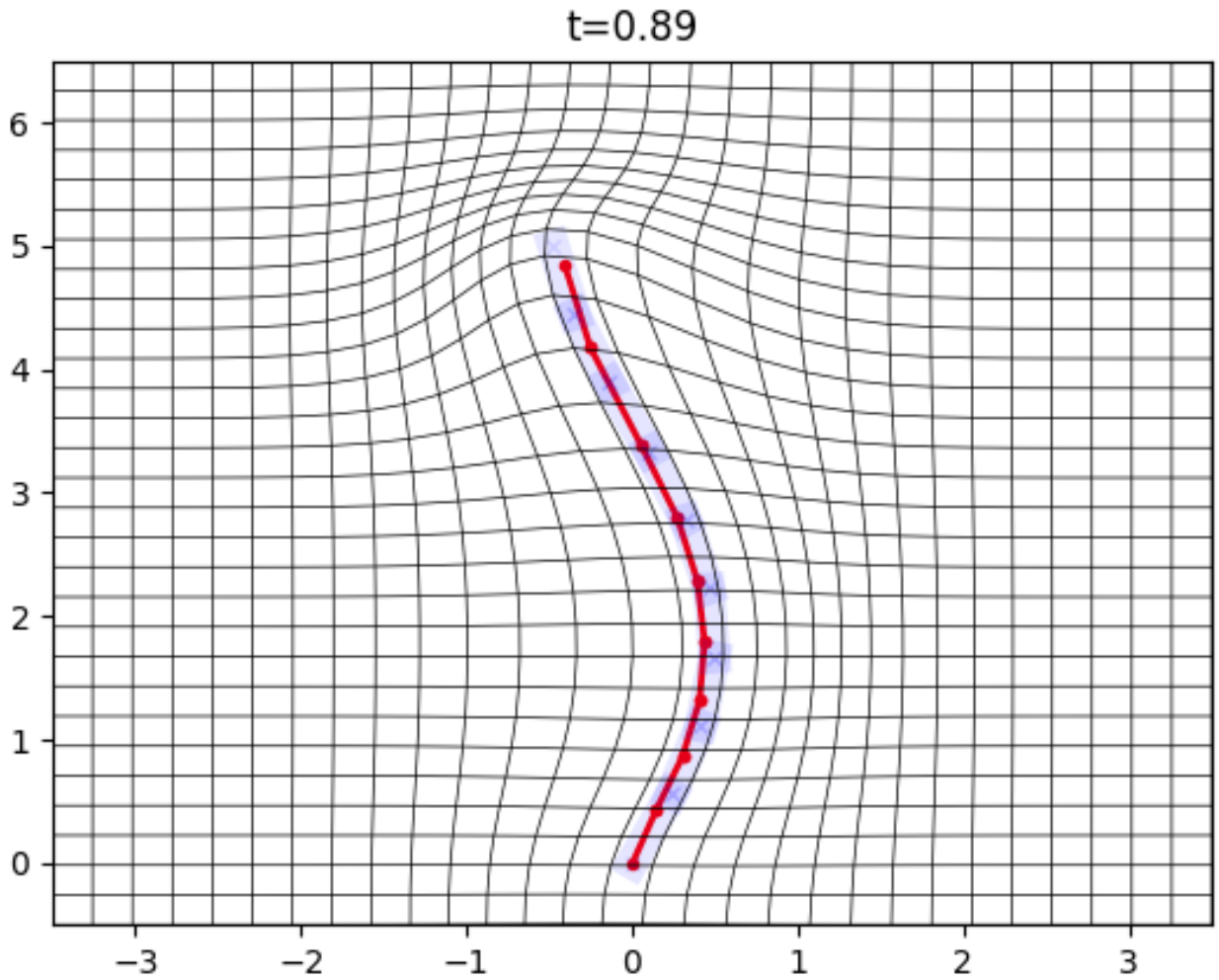


Shape deformation



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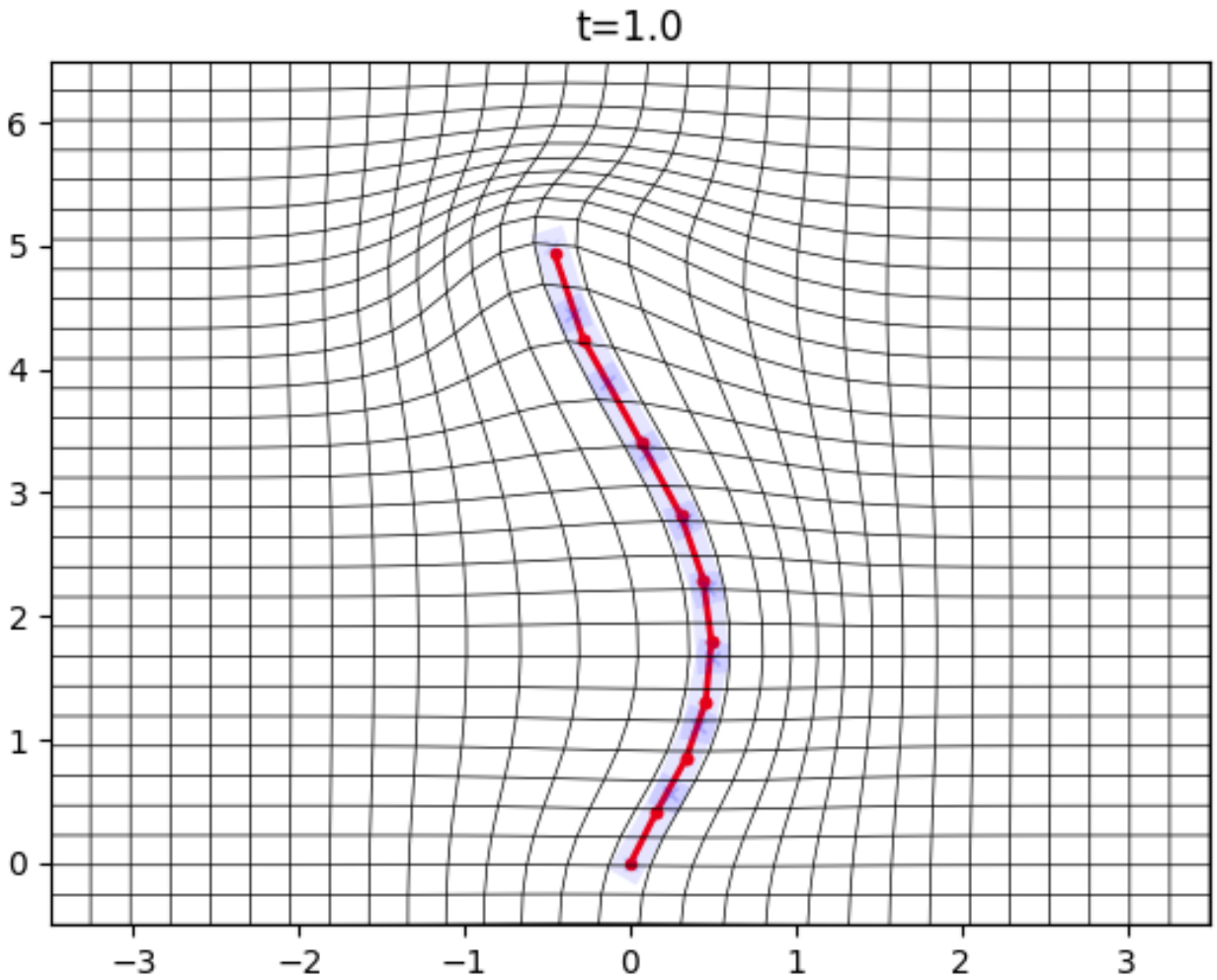


Solution of the variational problem

How to solve this variational problem ?

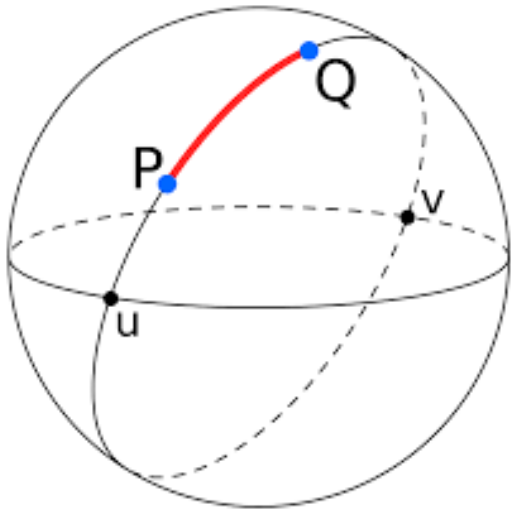
$$\min_{v \in L^2([0,1],V)} E(v) = \int_0^1 \frac{1}{2}|v_t|_V^2 \, dt + D(\varphi_1 \cdot q^{(0)}, q^{(1)}) \quad \text{s.t.} \quad \dot{q}_t = v_t \cdot q_t$$

Optimal control approach:

- Describe the geodesics through Hamiltonian equations
- Find the shortest path among geodesics using geodesic shooting

Geodesics and shortest paths

- Geodesics are generalization of straight lines to manifold. They are locally minimizing.
- The shortest path belongs to geodesics.
- Intuition : To find minima of a functional J, we first compute the critical points $dJ(x) = 0$, then minima belong to critical points.



Hamiltonian equations

Define the following Hamiltonian for $(q, p) \in T^*Q$ and $v \in V$:

$$H(q, p, v) = (p|v \cdot q) - \frac{1}{2}|v|_V^2$$

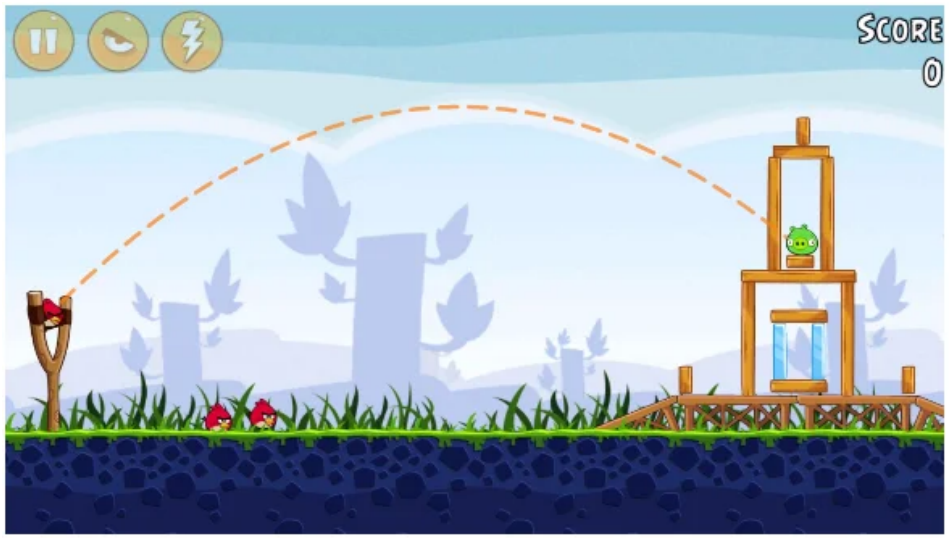
Geodesics are described by the Hamiltonian equations:

$$\begin{cases} \dot{q}_t = \partial_p H(q_t, p_t, v_t) \\ \dot{p}_t = -\partial_q H(q_t, p_t, v_t) \\ 0 = \partial_v H(q_t, p_t, v_t) \end{cases} \iff \begin{cases} \dot{q}_t = v_t \cdot q_t \\ \dot{p}_t = -(\partial_q \xi_{q_t}(v_t))^* p_t \\ v_t = K_V \xi_{q_t}^* p_t \end{cases}$$

Geodesic shooting

The variational problem simplifies to:

$$\min_{p_0 \in T^*_{q_0} Q} E(p_0) = \int_0^1 \frac{1}{2} |v_t|_V^2 \, dt + D(\varphi_1 \cdot q^{(0)}, q^{(1)}) \quad \text{s.t} \begin{cases} \dot{q}_t = v_t \cdot q_t \\ \dot{p}_t = -(\partial_q \xi_{q_t}(v_t))^* p_t \\ v_t = K_V \xi^*_{q_t} p_t \end{cases}$$

Algorithm :

- Start with $p_0 = 0$
- Shoot to get q_1
- Measure the distance $\|q_1 - q^{(1)}\|$
- Gradient descent to correct p_0