Decorrelation of vector fields with first variation of varifolds

Shape seminar

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LDDMM (Beg, Miller, Trouvé, Younes 2005)

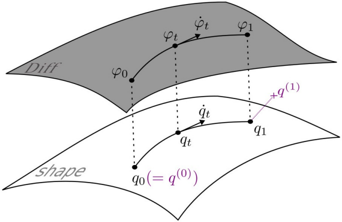
Let $v \in L^2([0,1],V)$ be a time-varying vector field where $V \hookrightarrow \mathcal{C}_0^2(\mathbb{R}^d,\mathbb{R}^d)$.
The flow of diffeomorphism φ^v generated by v is the unique solution of :

$$\dot{\varphi}_t^v = v_t \circ \varphi_t^v \qquad \text{s.t} \quad \varphi_0^v = \mathrm{id}$$

Shape registration corresponds to the following energy minimization problem :

$$\min_{v \in L^2([0,1],V)} E(v) = \int_0^1 \frac{1}{2}|v_t|_V^2 \, dt + D(\varphi_1 \cdot q^{(0)}, q^{(1)})$$

$$\text{s.t} \quad \dot{q}_t = v_t \cdot q_t \text{ and } q_0 = q^{(0)}$$



Coupling two types of deformations

Let $(w, v) \in L^2([0,1], W \times V)$ be two vector fields and ψ defined by :

$$\dot{\psi}_t = (w_t + v_t) \cdot \psi_t \qquad \text{s.t} \qquad \psi_0 = \mathrm{id}$$

- Dynamic of $q_t = \psi_t(q^{(0)})$:

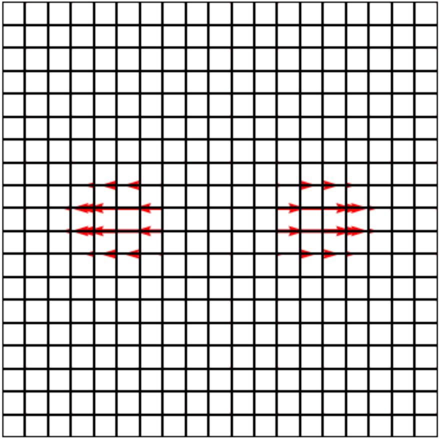
$$\dot{q}_t = w_t \cdot q_t + v_t \cdot q_t$$

- Shape registration :

$$\min_{(w,v) \in U \subset L^2([0,1], W \times V)} E(w, v) = \int_0^1 \mathrm{Cost}(w_t, v_t) dt + \mathcal{A}(q_1)$$

where $\mathcal{A} : Q \to \mathbb{R}$ is a data attachment term

Decorrelation with respect to a shape



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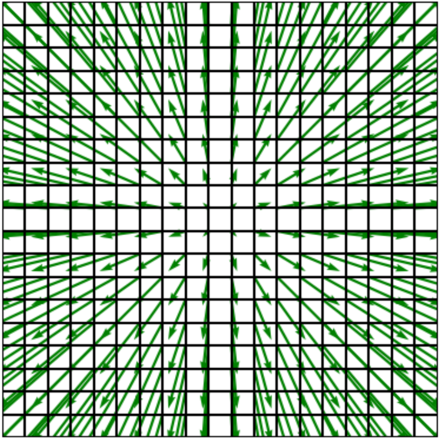

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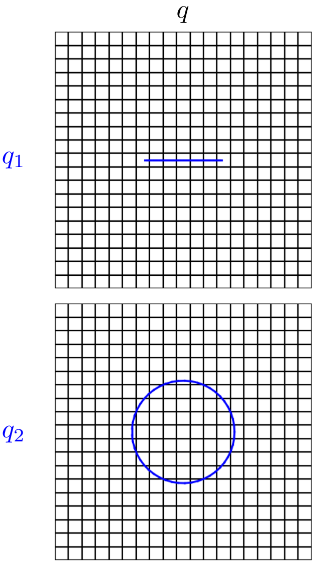

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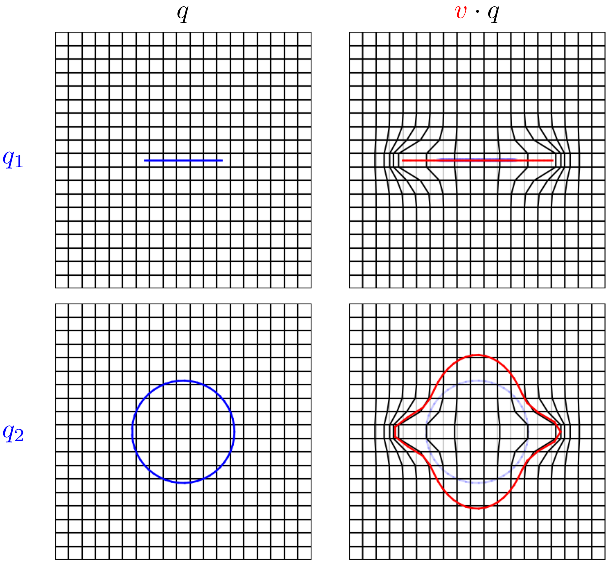

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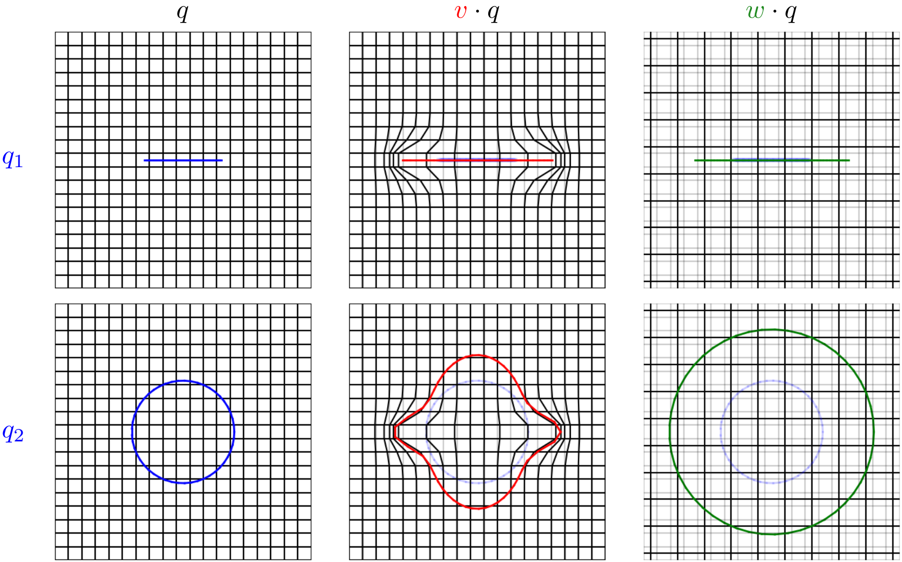


Correlation with respect to a shape

We define the correlation with respect to a shape q between a vector field $v \in V$ and a space of vector fields W by

$$\mathrm{Corr}_q(v, W) = \|w^*\|_W$$

where

$$w^* = \underset{w \in W}{\operatorname{argmin}} \|\delta\mu_q(v) - \delta\mu_q(w)\|^2_{\mathcal{W}'} + \lambda\|w\|^2_W$$

and $\mathcal{W} \hookrightarrow C^1_0(\mathbb{R}^d, \mathbb{R})$ is a Reproducing Kernel Hilbert Space

Varifold

Definition

A varifold is a continuous linear form on $\Omega = \{\omega : \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}\}$.
The varifold μ_q associated to the shape $q : X \to \mathbb{R}^d$ is defined by :

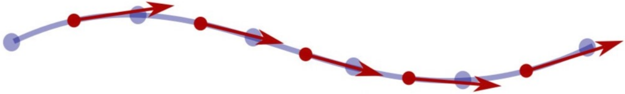
$$\mu_q(\omega) = \int_X \omega(x, \vec{t}(x))\, dx$$

where $\vec{t}$ represents a tangent/normal vector to the curve/surface.

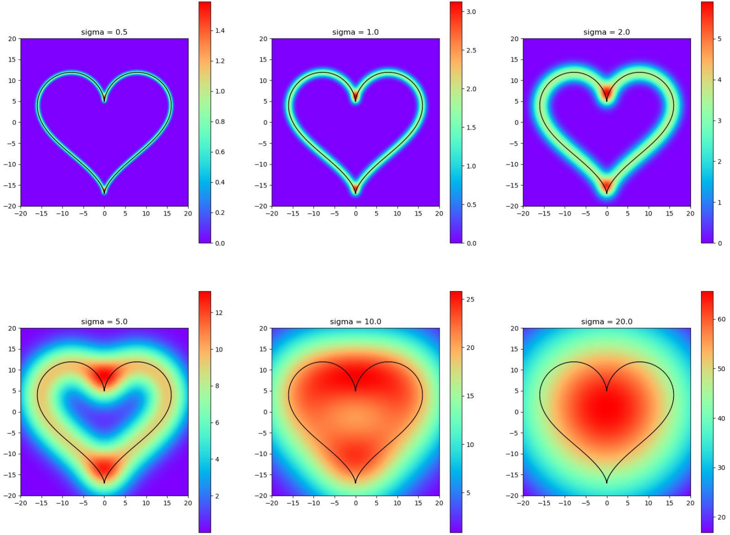
A discrete curve can be modeled by a varifold

$$\mu_q(\omega) = \sum_{(f^1, f^2) \in F} \|q_{f^2} - q_{f^1}\|_{\mathbb{R}^d}\ \omega(c(q_f), \vec{t}(q_f))$$

where $c(q_f) = \frac{q_{f^1} + q_{f^2}}{2}$ and $\overrightarrow{t}(q_f) = \frac{q_{f^2} - q_{f^1}}{\|q_{f^2} - q_{f^1}\|_{\mathbb{R}^d}}$.



Representation of a varifold with a gaussian kernel



Properties

Proposition

Given a RKHS $\mathcal{W} \hookrightarrow C_0^0(\mathbb{R}^d \times \mathbb{S}^{d-1})$ generated by a kernel $k_{\mathcal{W}} = k_E \otimes k_T$ and two curves q_a and q_b represented by $\mu_{q_a}, \mu_{q_b} \in \mathcal{W}'$, there exists a scalar product $\langle \mu_{q_a}, \mu_{q_b} \rangle_{\mathcal{W}'}$.

In the following, we will assume $k_T = 1$.

Proposition

The action of a diffeomorphism on a varifold is defined by

$$(\phi_* \mu_q)(\omega) = \mu_{\phi(q)}(\omega) = \sum_{(f^1, f^2) \in F} \|\phi(q_{f^2}) - \phi(q_{f^1})\|_{\mathbb{R}^d} \, \omega\left(c(\phi(q_f))\right)$$

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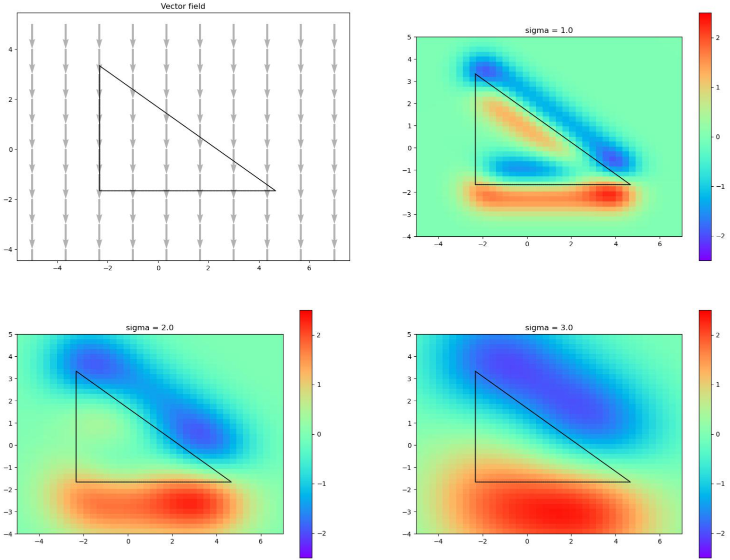
$\frac{d}{dt}\Big|_{t=0} \phi_t \cdot q = v \cdot q$

$\hookrightarrow \frac{d}{dt}\Big|_{t=0} (\phi_{t*}\mu_q) =: \delta\mu_q(v)$: 1st variation of a varifold

First variation of a varifold : translation



First variation of a varifold induced by a vector field

Theorem (Charon, Trouvé 2013)

Let $t \mapsto \phi_t$ be a flow of diffeomorphism such that $\phi_0 = \mathrm{id}$ and $\dot{\phi}_t|_{t=0} = v$. For $\omega \in C_0^1(\mathbb{R}^d, \mathbb{R})$,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \mu_{\phi_t(q)}(\omega) &= \sum_{(f^1,f^2)\in F} \left\langle v(q_{f^2}) - v(q_{f^1}), \frac{q_{f^1} - q_{f^2}}{\|q_{f^1} - q_{f^2}\|} \right\rangle \omega(c(q_f)) \\ &+ \|q_{f^1} - q_{f^2}\| \ \langle \nabla_x \omega(c(q_f)), c(v(q_f))\rangle\end{aligned}$$

Notation : $\delta\mu_q(v) := \frac{d}{dt}\Big|_{t=0} \mu_{\phi_t(q)}$

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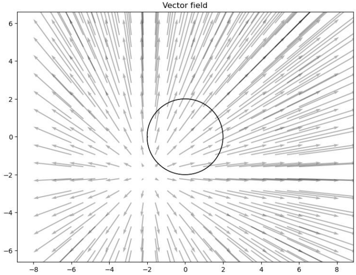
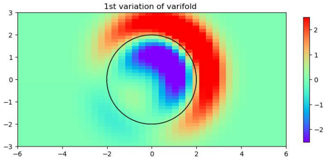
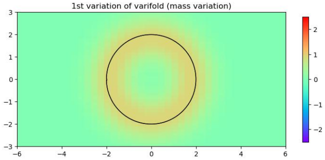
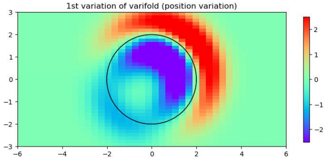
Decomposition of a varifold

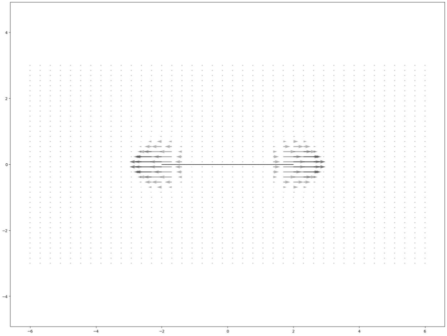




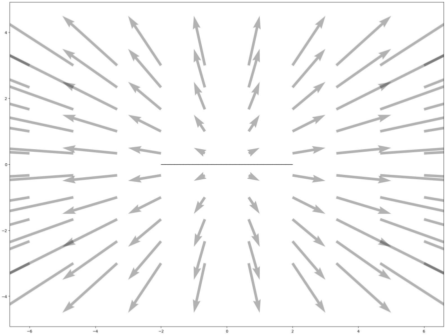




Influence of the shape



v



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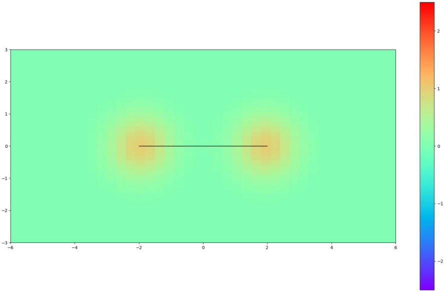
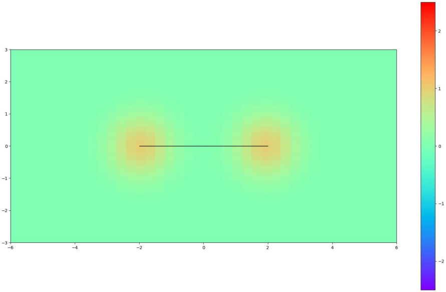


$K_{\mathcal{W}}\delta\mu_q(v)$



$K_{\mathcal{W}}\delta\mu_q(w)$

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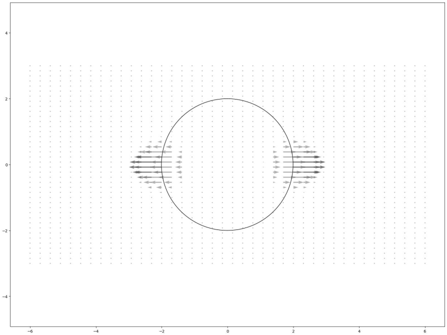

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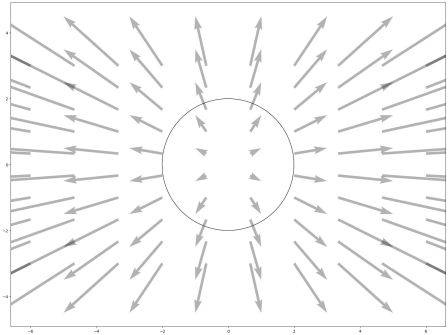

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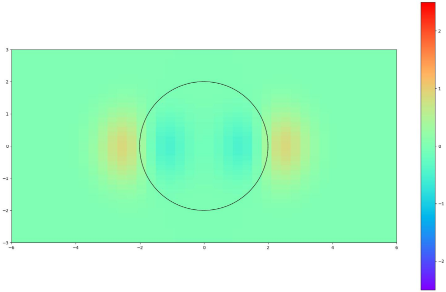

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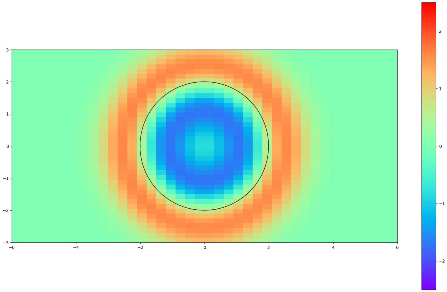

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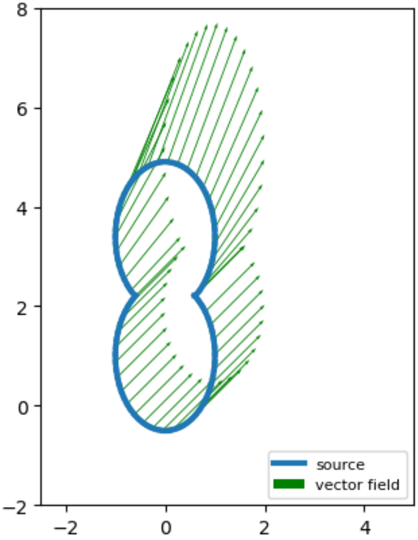
Influence of σ : $k(r) = e^{-r^2/\sigma^2}$



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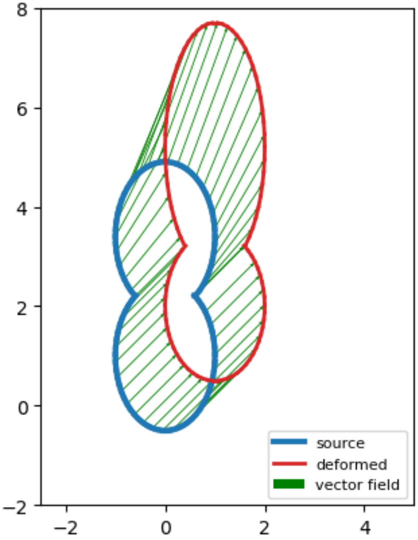

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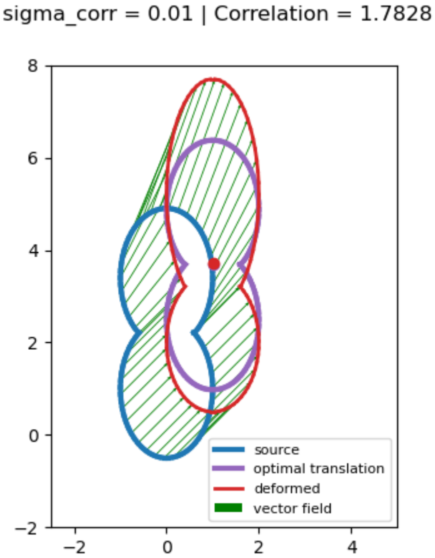

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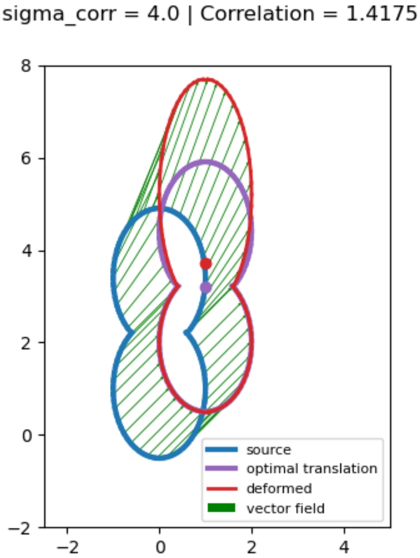

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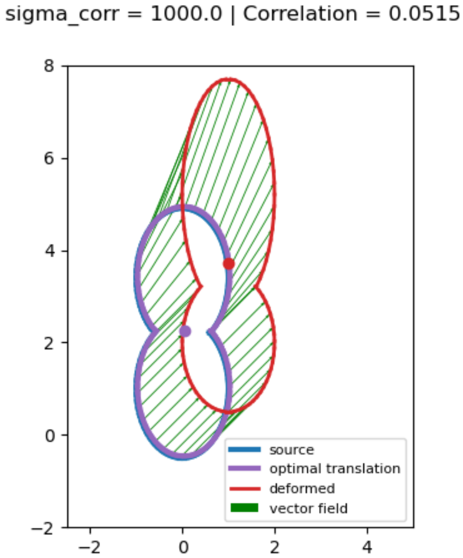

Dynamic generated by two vector fields

Given W and V two spaces of vector fields, we are interested in the following matching task :

$$\begin{aligned} \min_{(w,v)\in U\subset L^2([0,1],W\times V)} E(w,v) &= \int_0^1 \mathrm{Cost}(w_t,v_t)dt + \mathcal{A}(q_1) \\ \text{s.t} \quad & \dot{q}_t = w_t \cdot q_t + v_t \cdot q_t \end{aligned}$$

where $\mathcal{A} : \mathcal{Q} \to \mathbb{R}$ is a data attachment term.

Different approaches in the litterature :

- Multiscale kernel bundle, sum of gaussian kernel : Sommer et al. 2013, Risser 2011
- Semidirect product : Bruveris et al. 2012
- Hierarchical model : Pierron and Trouvé, 2024

Modelisation of the dynamic

- $E_1(w, v) = \int_0^1 \frac{1}{2}|w_t|_W^2 + \frac{1}{2}|v_t|_V^2 dt + \mathcal{A}(q_1)$

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- $E_2(w,v) = \int_0^1 \frac{1}{2}|w_t|_W^2 + \frac{1}{2}|v_t|_V^2 dt + \gamma \int_0^1 \frac{1}{2}\,\mathrm{Corr}_{q_t}^2(v_t, W) dt + \mathcal{A}(q_1)$

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Two models of admissible trajectories $U \subset L^2([0,1], W \times V)$:

- Geodesics associated to $w \mapsto E_1(w,v)$ and $v \mapsto E_1(w,v)$ (direct model).
- Geodesics resulting from a rewriting of the data attachment term (semidirect model).

Direct model

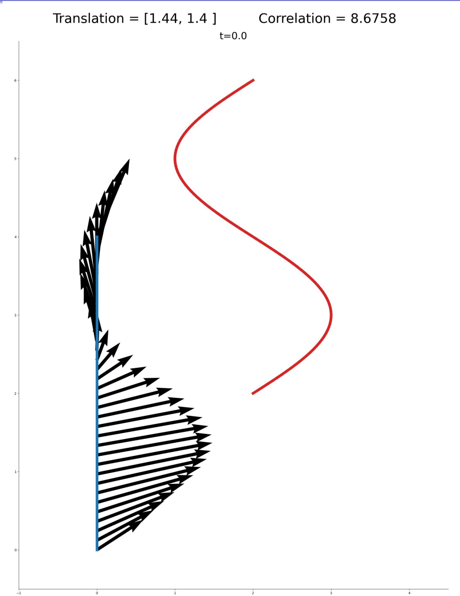
Considering the partial gradients of $E_1(w,v)$, we define a new problem.

$$\min_{p_0^W, p_0^V} E(p_0^W, p_0^V) = \int_0^1 \frac{1}{2}|v_t|_V^2 + \frac{1}{2}|w_t|_W^2 dt + \gamma \int_0^1 \frac{1}{2}\operatorname{Corr}^2_{q_t}(v_t, W)dt + \mathcal{A}(q_1)$$

$$\text{s.t} \quad \begin{cases} \dot{q}_t & = v_t \cdot q_t + w_t \cdot q_t \\ \dot{p}_t^W & = -(\partial_q(\xi_{q_t}^W(w_t) + \xi_{q_t}^V(v_t)))^* p_t^W \\ \dot{p}_t^V & = -(\partial_q(\xi_{q_t}^W(w_t) + \xi_{q_t}^V(v_t)))^* p_t^V \\ w_t & = K_W \xi_{q_t}^{W*} p_t^W \\ v_t & = K_V \xi_{q_t}^{V*} p_t^V \end{cases}$$

where $\xi_{q_t}^W(w_t) = w_t \cdot q_t$, $\xi_{q_t}^V(v_t) = v_t \cdot q_t$ and $(p_t^W, p_t^V) \in T_{q_t}^*Q \times T_{q_t}^*Q$.

Direct model without decorrelation



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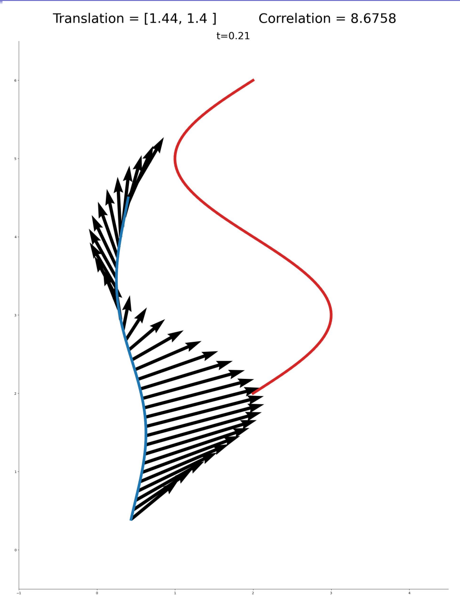

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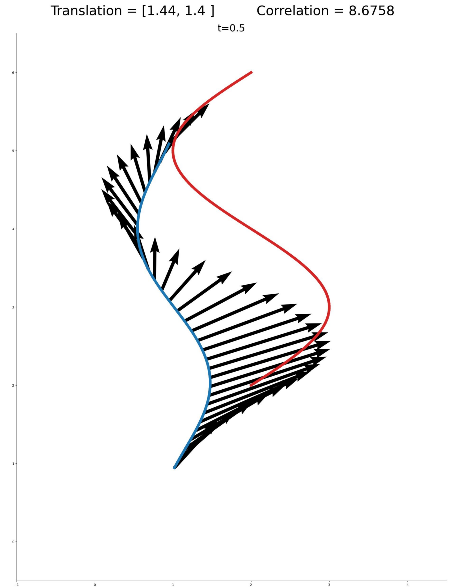

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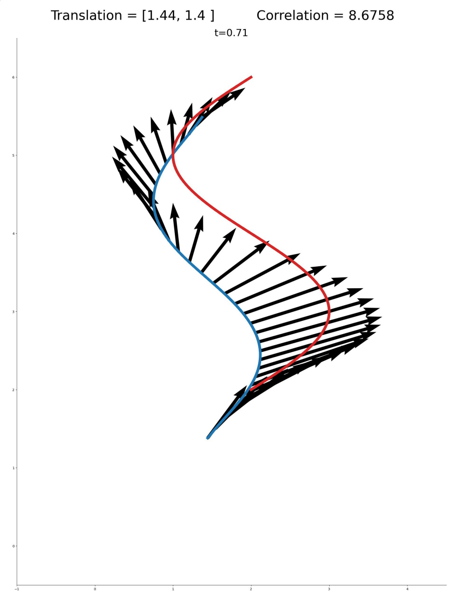

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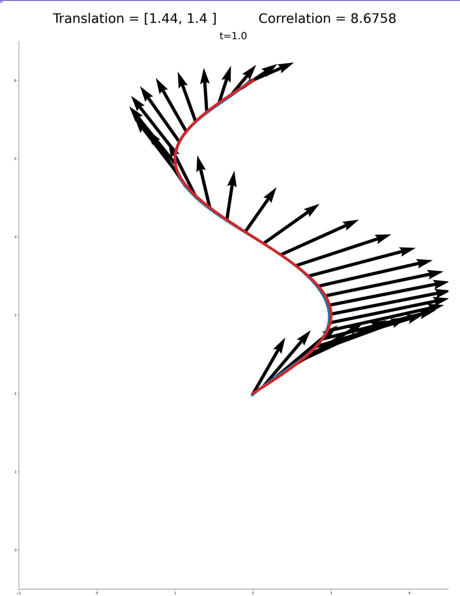

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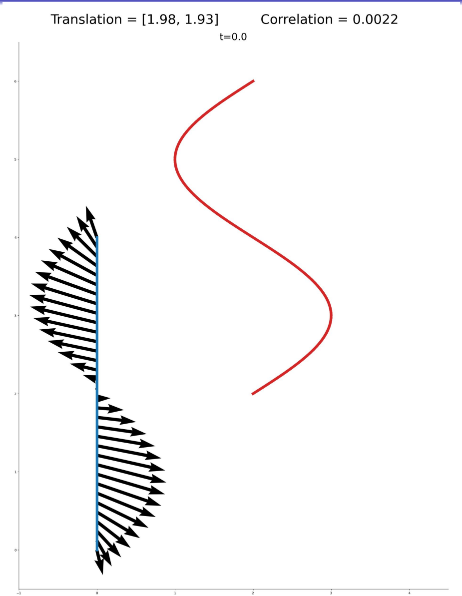

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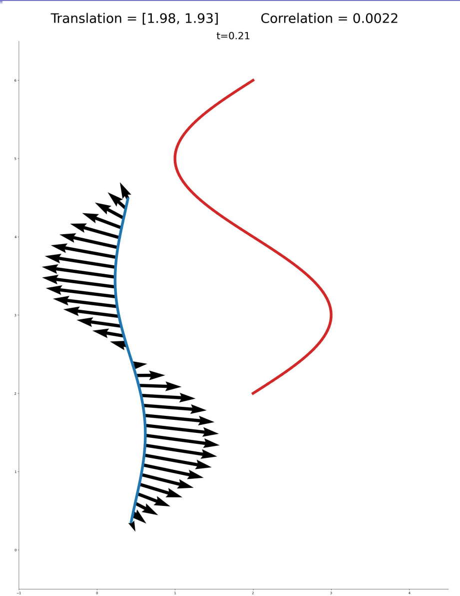

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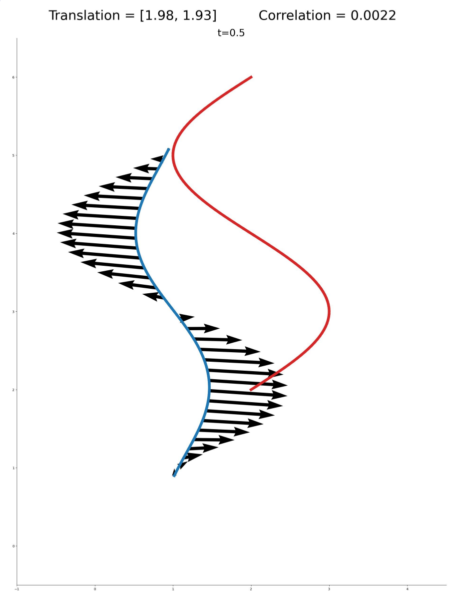
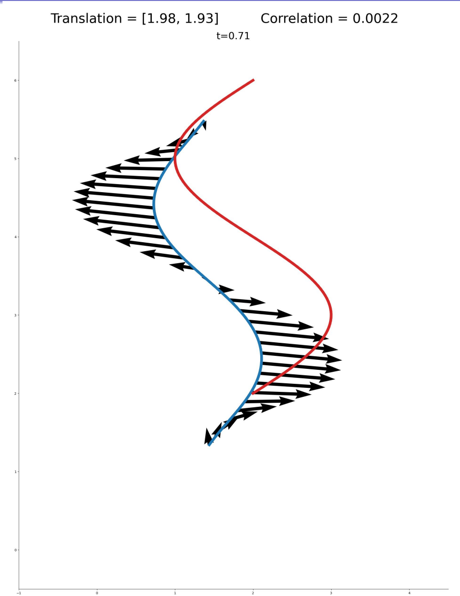
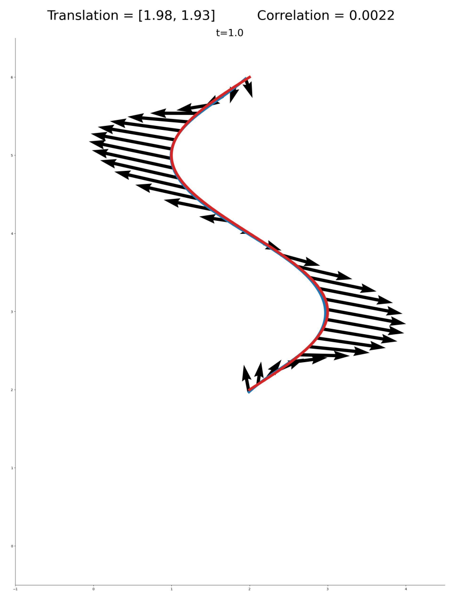


Direct model with decorrelation



Direct model with decorrelation



Augmented shape space

- Another approach : Extent the shape space to $G \times Q$ where G is a finite-dimensional group of deformations (e.g isometries).

$$E_2(w,v) = \int_0^1 \frac{1}{2}|w_t|_W^2 + \frac{1}{2}|v_t|_V^2 dt + \gamma \int_0^1 \frac{1}{2}\,\mathrm{Corr}_{q_t}^2(v_t, W)dt + \mathcal{A}(g_1, \tilde{q}_1)$$

- New shape : $(g, \tilde{q}) = (g, g^{-1} \cdot q)$.

- Rewriting of the data attachment term :
$\tilde{\mathcal{A}}(g, \tilde{q}) = \mathcal{A}(g \cdot \tilde{q}) = \mathcal{A}(q) \Longrightarrow p^{\mathfrak{g}} \in T_g^* G$ and $\tilde{p} \in T_{\tilde{q}}^* Q$.

Semidirect model (joint work with Thomas Pierron)

Let G be a finite dimensional Lie group and $\mathfrak{g}$ its Lie algebra.

$\rightarrow$ Assumptions :

- G acts on $\mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ via $\alpha_g(\varphi)$.
- $G \ltimes \mathrm{Diff}_{C_0^k}(\mathbb{R}^d)$ acts on $G \times Q$: $(g,\varphi)\cdot(h,q) = (gh, g\cdot(\varphi\cdot q))$

Example : If $G = \mathrm{SO}_d(\mathbb{R})$, then $\alpha_R(\varphi)(x) = R^{-1}\varphi(Rx)$ and $(R,\varphi)\cdot q = R\varphi(q)$.

Semidirect model

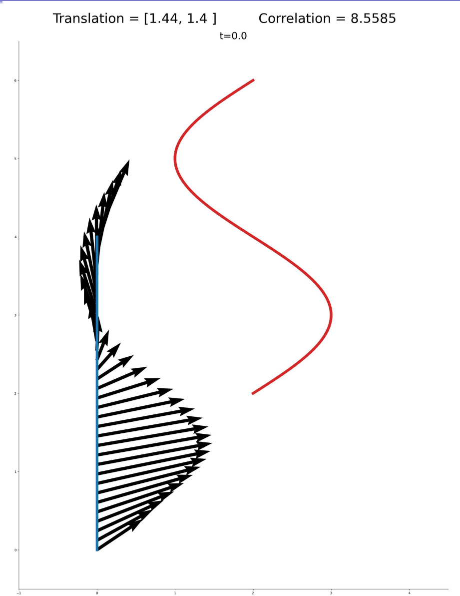
- New shape : $(g, \tilde{q}) = (g, g^{-1}q)$
- New data attachment term : $\tilde{\mathcal{A}}(g, \tilde{q}) = \mathcal{A}(g\tilde{q})$

$$\min_{(p_0^{\mathfrak{g}}, \tilde{p}_0)} E(p_0^{\mathfrak{g}}, \tilde{p}_0) = \int_0^1 \frac{1}{2}|v_t|_V^2 + \frac{1}{2}|X_t|_{\mathfrak{g}}^2 dt + \gamma \int_0^1 \frac{1}{2} \operatorname{Corr}^2_{q_t}(v_t, \mathfrak{g}) dt + \tilde{\mathcal{A}}(g_1, \tilde{q}_1)$$

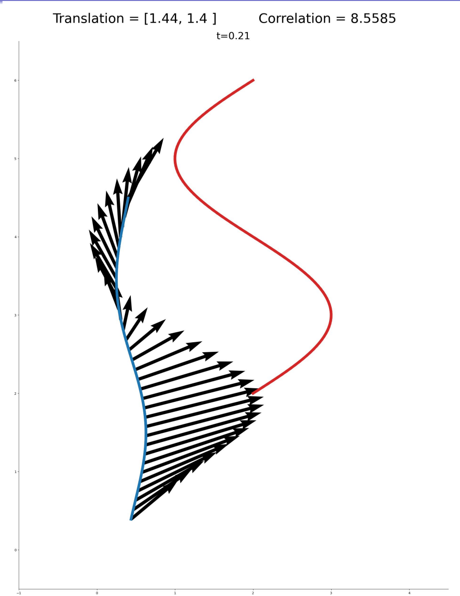
$$\text{s.t} \quad \begin{cases} \dot{g}_t &= X_t \cdot g_t \\ \dot{\tilde{q}}_t &= d_{\mathrm{id}}\alpha_{g_t}(v_t) \cdot \tilde{q}_t \\ \dot{p}_t^{\mathfrak{g}} &= -(\partial_g \xi_{g_t}^{\mathfrak{g}}(X_t))^* p_t^{\mathfrak{g}} - (\partial_g \xi_{\tilde{q}_t}^{V}(d_{\mathrm{id}}\alpha_{g_t}(v_t)))^* \tilde{p}_t \\ \dot{\tilde{p}}_t &= -(\partial_q \xi_{\tilde{q}_t}^{V}(d_{\mathrm{id}}\alpha_{g_t}(v_t))^* \tilde{p}_t \\ X_t &= K_{\mathfrak{g}} \xi_{g_t}^{\mathfrak{g}*} p_t^{\mathfrak{g}} \\ v_t &= K_V \xi_{q_t}^{V*} \partial_q A^G(g_t^{-1}, q_t)^* \tilde{p}_t \end{cases}$$

where $\xi_{g_t}^{\mathfrak{g}}(X_t) = X_t g_t$ and $(p_t^{\mathfrak{g}}, \tilde{p}_t) \in T_{g_t}^* G \times T_{\tilde{q}_t}^* Q$:

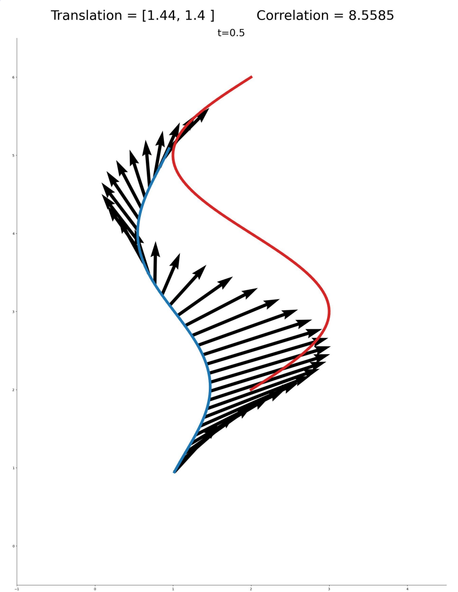
Semidirect model without decorrelation



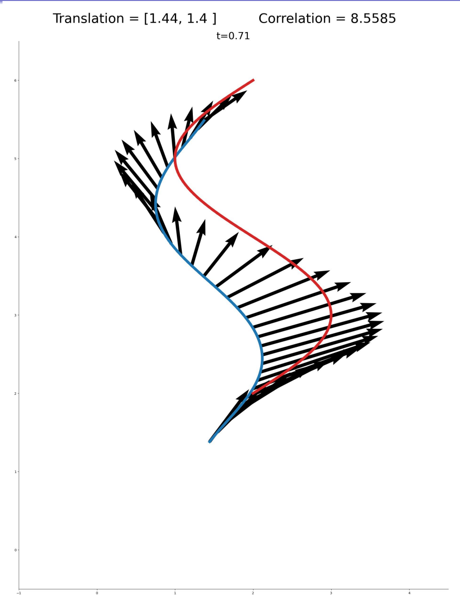
Semidirect model without decorrelation



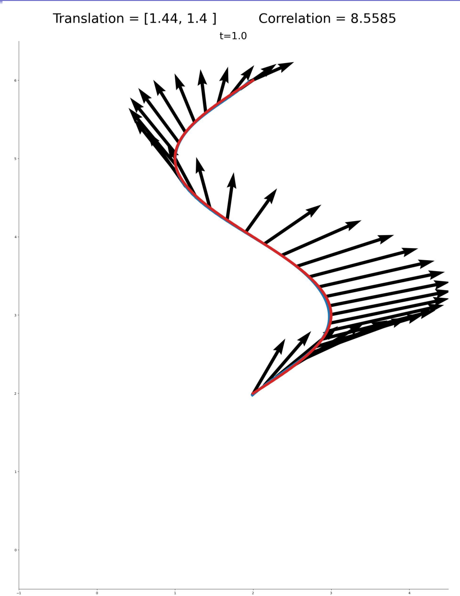
Semidirect model without decorrelation



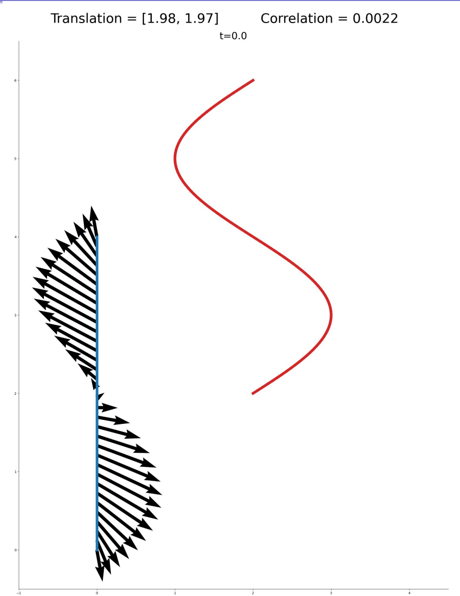
Semidirect model without decorrelation



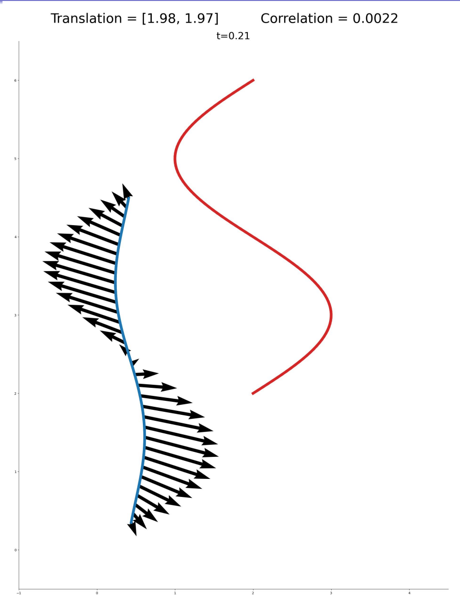
Semidirect model without decorrelation



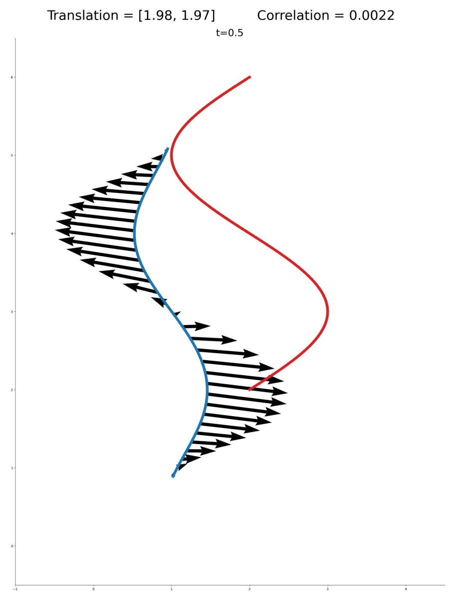
Semidirect with decorrelation



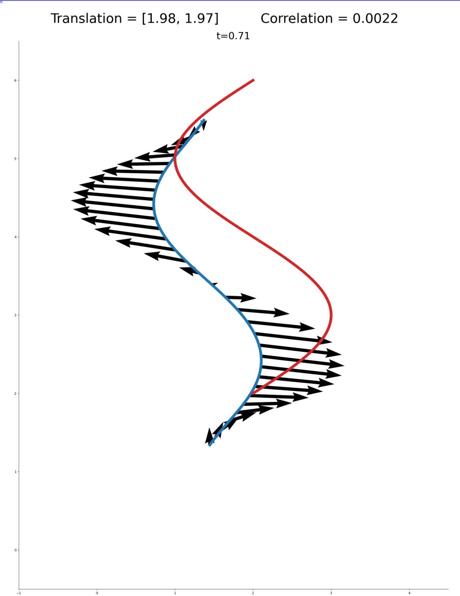
Semidirect with decorrelation



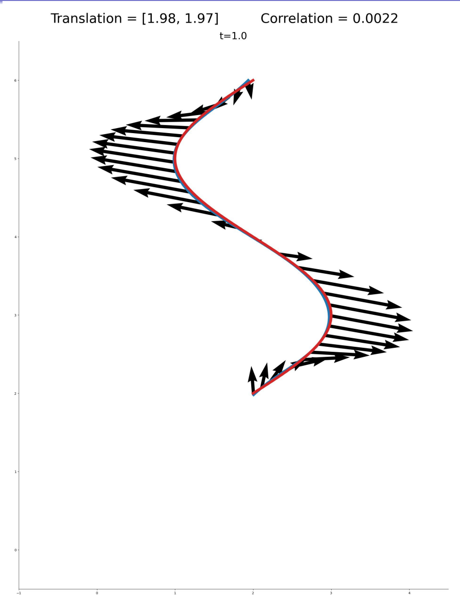
Semidirect with decorrelation



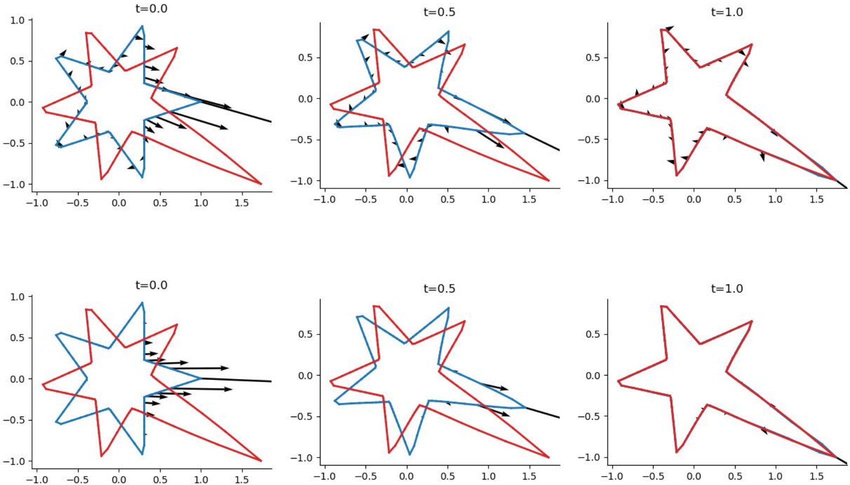
Semidirect with decorrelation



Semidirect with decorrelation



Decorrelation from the space of rotations



Comparison of the geodesics

Semidirect model :

Comparison of the geodesics

Semidirect model :

- Shape (g, q)
  $\rightarrow (\dot{g}_t, \dot{q}_t) = (X_t g_t, X_t \cdot q_t + v_t \cdot q_t)$
  $\rightarrow \begin{cases} X_t & = K_{\mathfrak{g}} \xi_{g_t}^{\mathfrak{g}*} p_t \\ v_t & = K_V \xi_{q_t}^{V*} p_t \end{cases}$ with $p_t \in T_{q_t}^* Q$

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- Shape $(g, \tilde{q}) = (g, g^{-1} q)$
  $\rightarrow (\dot{g}_t, \dot{\tilde{q}}_t) = (X_t g_t, \partial_q A^G(g_t^{-1}, q_t) \xi_{q_t}^V (v_t))$
  $\rightarrow \begin{cases} X_t & = K_{\mathfrak{g}} \xi_{g_t}^{\mathfrak{g}*} p_t^{\mathfrak{g}} \\ v_t & = K_V \xi_{q_t}^{V*} \partial_q A^G(g_t^{-1}, q_t)^* \tilde{p}_t \end{cases}$ with $p_t^{\mathfrak{g}} \in T^*_{g_t} G$, with $\tilde{p}_t \in T^*_{\tilde{q}_t} Q$

Comparison of the geodesics

Semidirect model :

- Shape (g, q)
  $\rightarrow (\dot{g}_t, \dot{q}_t) = (X_t g_t, X_t \cdot q_t + v_t \cdot q_t)$
  $\rightarrow \begin{cases} X_t &= K_{\mathfrak{g}} \xi^{\mathfrak{g}*}_{g_t} p_t \\ v_t &= K_V \xi^{V*}_{q_t} p_t \end{cases}$ with $p_t \in T^*_{q_t} Q$

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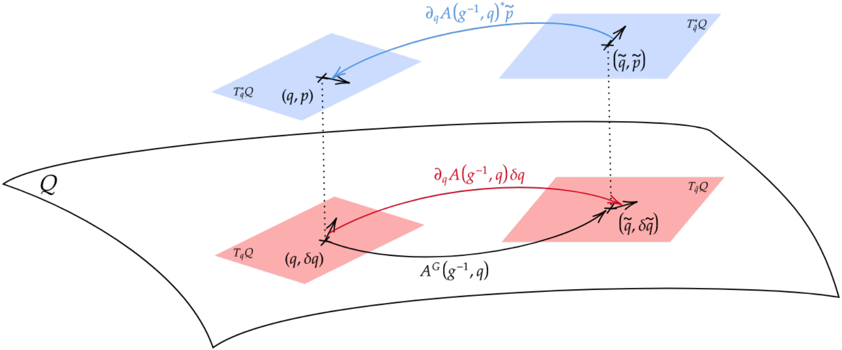
  $\rightarrow (\dot{g}_t, \dot{\tilde{q}}_t) = (X_t g_t, \partial_q A^G(g_t^{-1}, q_t) \xi^V_{q_t}(v_t))$
  $\rightarrow \begin{cases} X_t &= K_{\mathfrak{g}} \xi^{\mathfrak{g}*}_{g_t} p^{\mathfrak{g}}_t \\ v_t &= K_V \xi^{V*}_{q_t} \partial_q A^G(g_t^{-1}, q_t)^* \tilde{p}_t \end{cases}$ with $p^{\mathfrak{g}}_t \in T^*_{g_t} G$, with $\tilde{p}_t \in T^*_{\tilde{q}_t} Q$

Proposition

If $p_0 = \tilde{p}_0$, then $p_t = \partial_q A^G(g_t^{-1}, q_t)^* \tilde{p}_t$ for every $t \in [0, 1]$.

Comparison of the geodesics

Relation between T_q^*Q and $T_{g^{-1}q}^*Q$ corresponds to the lift of $q \mapsto A^G(g^{-1}, q)$ on T^*Q.



Comparison of the models

$$E_2(w,v) = \int_0^1 \frac{1}{2}|w_t|_W^2 + \frac{1}{2}|v_t|_V^2 \, dt + \gamma \int_0^1 \frac{1}{2} \operatorname{Corr}_{q_t}^2(v_t, W) \, dt + \mathcal{A}(q_1)$$

Direct model	**Semidirect model**
$v = K_V \xi_q^{V*} p^V$	$v = K_V \xi_q^{V*} \partial_q A(g^{-1}, q)^* \tilde{p}$
$w = K_W \xi_q^{W*} p^W$	$X = K_{\mathfrak{g}} \xi_g^{\mathfrak{g}*} p^{\mathfrak{g}}$

Comparison of the models

$$E_2(w,v) = \int_0^1 \frac{1}{2}|w_t|_W^2 + \frac{1}{2}|v_t|_V^2 \, dt + \gamma \int_0^1 \frac{1}{2} \operatorname{Corr}_{q_t}^2(v_t, W) \, dt + \mathcal{A}(q_1)$$

Direct model	**Semidirect model**
$v = K_V \xi_q^{V*} p^V$ $w = K_W \xi_q^{W*} p^W$	$v = K_V \xi_q^{V*} \partial_q A(g^{-1}, q)^* \tilde{p}$ $X = K_{\mathfrak{g}} \xi_g^{\mathfrak{g}*} p^{\mathfrak{g}}$
any deformations	isometries + scaling

Comparison of the models

$$E_2(w,v) = \int_0^1 \frac{1}{2}|w_t|_W^2 + \frac{1}{2}|v_t|_V^2\,dt + \gamma\int_0^1 \frac{1}{2}\,\mathrm{Corr}^2_{q_t}(v_t,W)\,dt + \mathcal{A}(q_1)$$

Direct model	**Semidirect model**
$v = K_V\xi_q^{V*}p^V$ $w = K_W\xi_q^{W*}p^W$	$v = K_V\xi_q^{V*}\partial_q A(g^{-1},q)^*\tilde{p}$ $X = K_{\mathfrak{g}}\xi_g^{\mathfrak{g}*}p^{\mathfrak{g}}$
any deformations $p^W \in T^*Q$	isometries + scaling $p^{\mathfrak{g}} \in T^*G$

To take-home

- 1st variation of a varifold : $\delta\mu_q(v) := \frac{d}{dt}\Big|_{t=0}(\varphi_{t*}\mu_q) \in C_0^1(\mathbb{R}^d, \mathbb{R})'$
- $\mathrm{Corr}_q(v, W) = \|w^*\|_W$
  where $w^* = \mathrm{argmin}_{w\in W} \|\delta\mu_q(v) - \delta\mu_q(w)\|^2_{\mathcal{W}'} + \lambda\|w\|^2_W$
- Dynamic generated by W and V :

$$E_2(w, v) = \int_0^1 \frac{1}{2}|w_t|^2_W + \frac{1}{2}|v_t|^2_V\, dt + \gamma \int_0^1 \frac{1}{2}\,\mathrm{Corr}^2_{q_t}(v_t, W)\, dt + \mathcal{A}(q_1)$$

  - Direct model : Partial gradients of E_1
  - Semidirect model : Extension of the data attachment term
    $\tilde{\mathcal{A}}(g, \tilde{q}) = \mathcal{A}(g\tilde{q}) = \mathcal{A}(q)$

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